Mathematics of Planet Earth

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FOREWORD

Reaching 50 is a milestone worth celebrating and in 2013 we are doing just that at the Mathematical Association of Victoria Annual Conference. Over the past 50 years, the MAV Annual Conference has drawn together people with a passion for mathematics to share insights into teaching and research that can further our mathematics community. This 50th conference has the theme of The Mathematics of Planet Earth, chosen to showcase the International Year of the Mathematics of Planet Earth. Just as many events through this year have revealed a myriad of applications of mathematics (for example, see http://mathsofplanetearth.org.au/), the submissions to the conference publication demonstrate the wide range of areas of interest our Australasian participants hold within the field of mathematics.

As editors we have felt privileged to read the many submissions to the proceedings written by committed and talented mathematics educators and we thank the authors for their outstanding contributions. It has been a pleasure to work with them during this process.

As well as continuing to offer Peer-reviewed and Student papers, this year we take pleasure in re-introducing Double blind-reviewed papers to the conference proceedings. These papers have undergone a rigorous review process. A new category has been introduced to the publication – Summary submissions – which has accommodated those presenters who wish to share key points related to their presentation in a shortened format. We trust that all papers in the 2013 proceedings will be of value to readers and offer food for thought.

We also thank the contributors to the blind and peer reviewing process for their willingness to review and for their insightful and constructive comments. Their contribution to the continued success of the MAV conference proceedings is greatly appreciated.

We close by thanking the MAV conference staff for their professionalism and for the support provided to us in our role as editors.

Happy reading.

Andrea McDonough and Ann Downton (Australian Catholic University); Leicha A. Bragg (Deakin University) (Editors)

The Review Process for the Mathematical Association of Victoria 50th Annual Conference Proceedings

The Editors received 10 full papers for the Double blind review process, for which the identities of author and reviewer were concealed from each other. Details in the papers that identified the authors were removed to protect the review process from any potential bias, and the reviewers' reports were anonymous. Two reviewers reviewed each of the 10 blind review papers and if they had a differing outcome a third reviewer was required. Nine of the 10 papers were accepted for publication and one was withdrawn.

In addition, we received 15 full papers for the Peer review process, one Student paper, and 10 Summary papers. These papers were reviewed by a combination of external reviewers and the editorial team.

In total, 35 papers are published in the Mathematical Association of Victoria 50th Annual Conference Proceedings. A total of 21 reviewers assisted in the process, all of whom provided thoughtful feedback and were outstanding in responding quickly to our requests.

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MAXIMIZATION USING MAPLE: POLYA APPROACH WITH MULTIPLE REPRESENTATION – WITH OR WITHOUT CALCULUS

Bill Blyth

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Maple, a leading Computer Algebra System (CAS), has been used to support student learning for 20 years at RMIT University. We revisit some "Find the maximum" problems typical of VCE calculus using Maple to implement a detailed and structured Polya "How to Solve It" approach used by first year mathematics students at RMIT University.

As well as the calculus solution, we obtain the solution without calculus by exploration using visualization, animations and multiple representations – and prove that our proposed solution is correct (without calculus).

Finally, we briefly discuss our assessment experience with e-Marking and Computer Aided Assessment with Maple or MapleTA.

Introduction

Upper level school mathematics usually includes an introduction to calculus. Typically first year university science and engineering include mathematics courses with a large component being calculus which revises and extends school level calculus. Many studies have been undertaken at this secondary/tertiary interface and tend to highlight problems such as student inhomogeneous mathematical backgrounds and student negative attitudes towards mathematics and computing. One study of UK university students (Mann and Robinson 2009) reported "that 59% of students find their lectures boring half the time and 30% find most or all of their lectures boring." A range of different teaching methods was investigated. The study showed that students regarded lab sessions as the most boring and computer sessions as the second most boring. "Computer sessions too have the potential to be stimulating or tedious; the findings of this study suggest that too many fall into the later category. This could be due to the manner in which sessions are conducted (e.g. are the computer tasks relevant and interesting?), …".

Are these findings surprising? Most courses are strongly teacher/lecturer centered with student work mostly done without technology. Several countries, particularly Austria, have used CAS in the senior schools for decades. The State of Victoria in Australia introduced the VCE Mathematical Methods (CAS) course about a decade ago. Unfortunately typical use of CAS calculators, or professional level CAS such as Maple, is to complete exercises with little or no variation from the "by hand" approach. Development of innovative teaching materials which exploit the strengths of the CAS for tightly integrated visualization, symbolic and numerical computation and Computer Aided Assessment (CAA) can be used to produce active and engaged learners of mathematics and the technology at school and university.

Preliminaries

At RMIT University, we use Maple as our CAS. What follows may be adapted for use with other CAS devices or software although some parts, such as the multiple representations, may not be implementable with a CAS calculator. Students need to undertake very carefully prepared preliminary activities so that they can properly benefit from the proposed activity. At RMIT University, the first year students had weekly one hour Maple lab classes, with no lectures. They worked collaboratively in small groups of 2 to 4 students on a structured program including:

- an *Introduction to Maple* - 2 hours
- an *Introduction to Animation* - 1 hour
- an extended *Animation Assignment* - 3 hours
- *Spot the Curve* - 1 hour
- A *Polya How to Solve It* approach to finding maximum area – 1 hour
- the *Norman Window assignment* (maximizing the area) - 1 hour

The *Spot the Curve* activity provides randomly generated quadratic and cubic curves and translated curves such that, by using visualization, graphics and animations, students can explore and determine translation of curves. Their answers were automatically marked within Maple (see Blyth and Labovic 2008).

The activity *Polya, How to Solve It* (Polya 1945) provides the detailed steps required to find the maximum area of a rectangular block that a farmer can fence with a fixed length of fencing. The rectangle is fenced on three sides with a river being the fourth side.

Students then work on an assignment (with different parameterization for different groups) to solve the *Norman Window* problem (see Blyth 2012). Traditionally these problems are applications of calculus problems, but we have additional activities in which methods without calculus and multiple representations are used.

The Minimum Distance from a Point to a Line

The original problem is "Find the point on the straight line $x+2\,y=5$ which is closest to the origin." Parameterization for different student groups is simple here since this line is a particular case of $ax+by=c$.

Minimise Distance – Using Calculus

The traditional use of calculus is the first of four solution methods we present in this paper. The objective function (that is to be minimized) can be treated as a quadratic polynomial. This problem can be used in the middle school to provide an applied context to the algebra of completion of the square: and to also introduce (or reinforce) the Polya approach to problem solving. See Figure 1 (a) for a labelled diagram (which students can construct or be given).

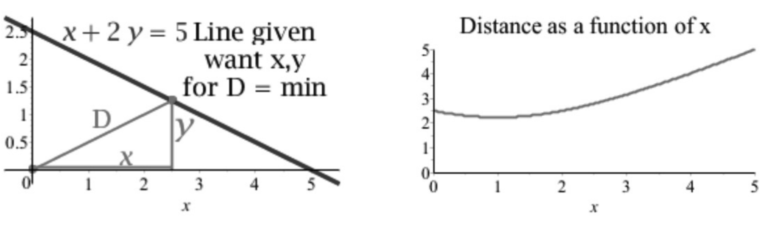

Figure 1. (a) A diagram for the problem: minimize the distance form the origin to the given line. (b) A plot of distance as a function of x only.

What is to be minimized? Let D be the distance from the given point, the origin, to the line $x + 2y = 5$. The problem is to minimise the distance, D. Since the distance must be positive it is easier if we avoid the square root by solving the equivalent problem: minimise

$D^2 = x^2 + y^2$. In Maple, D is a 'reserved word' used to denote the differentiation operator. Also, with CAS, it is not permitted to assign to a variable squared. In order to avoid any difficulties, we denote D^2 by DSq. As we need to write the expression for D^2 in terms of one variable only, we choose to eliminate y (since students are usually more comfortable with x as the independent variable).

Thus DSq = $5/4\,x^2 - 5/2\,x + 25/4$.

What is the range for the chosen variable? Obviously x must be real, but what else? By looking at the diagram and using our knowledge of geometry, the minimum D must be less than the value of the x axis intercept (where $x = xInter$) and the value of the y axis intercept (where $x = 0$ and $y = yInter$). Thus $0 < x < xInter$, where $xInter = c/a = 5$. Hence $0 < x < 5$.

Find the minimum using calculus. First we show that the derivative of DSq is $5/2\,x - 5/2$ and then solve the equation $5/2\,x - 5/2 = 0$ for x. So the x solution is $xSol = 1$. Substitution back into the equation of the line gives $ySol = 2$ and hence the minimum D is $\sqrt{5}$.

Is $x=xSol$ in the domain? Yes, we do have $0 < xSol < xInter$. Since calculus gives the RELATIVE extrema, we must check whether any of the endpoints of the domain gives the ABSOLUTE extrema.

When $x = 0, y = 5/2$, D = 5/2 and when $x = 5, y = 0$, D = 5.

Thus the relative minimum is also the (absolute) minimum. Since it is easy to plot functions using Maple, we could also view a graph of D^2 or D as a function of the one variable (see Figure 1 (b)).

State the solution to the given problem. Read the problem again and state the required solution: the minimum distance, D, from the origin to the line $x + 2y = 5$, is $\sqrt{5}$ when the point on the line has (x, y) coordinates (1, 2).

Minimize Distance – Without Calculus

A number of optimization problems in the standard textbooks have applied problems in which the objective function is a quadratic polynomial: for example, the *Farmer fencing a block of land* and the *Norman Window* problems, mentioned above. Having taught a very large number of first year calculus students related optimization problems including problems with a quadratic polynomial as the objective function, it's interesting to note that no student has noticed that calculus is not needed here.

Once it is noticed that D squared can be minimized, it is easy to obtain DSq as a function of x only as above.

DSq = $5/4\,x^2 - 5/2\,x + 25/4$.

A plot of DSq against x shows a parabolic shape (see Figure 2 (a)). School students in Victoria have been using graphics calculators, and more recently CAS calculators, for two decades. They have acquired a very useful skill: "zooming in" on interesting parts of a graph. Although this is generally ignored at university, it is useful, for example to find good starting values for Newton's method to obtain zeros of a function. A plot, having zoomed in near the obvious x value for a minimum, is shown in Figure 2 (b). It indicates that $x = 1$ is the likely solution.

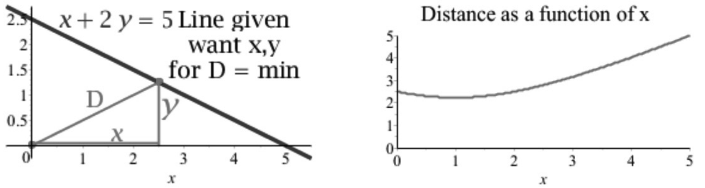

Figure 2. (a) A plot of distance squared and (b) the plot zoomed in about x=1.

The parabolic shape of the objective function, DSq(x), is particularly evident from the zoomed in plot!

Since the objective function is a quadratic polynomial, we can use the method of completing the square for quadratic polynomials to calculate the coordinates of the minimum point.

Now DSq is $5/4\,x^2 - 5/2\,x + 25/4 = 5/4\,(x^2 - 2x + 5) = 5/4\,((x-1)^2 - 1 + 5)$

That is, $\text{DSq} = 5/4\,(x-1)^2 + 5$.

The positive coefficient of the squared term implies that there exists a minimum, as can be seen from the graph. The minimum is DSq = 5 when x=1.

As in the previous method, the y value of the solution is found by substitution of x into the equation of the line. The solution of the problem can then be stated.

Comment. For students completing the square by hand, we want the algebra to be straightforward, so we ensure that the x value of the solution is a low integer or perhaps a half integer. The symbolic solution of this problem is, for $c > 0$:

$$x, y, \text{D} = \frac{c\,a}{a^2+b^2}, \frac{c\,b}{a^2+b^2}, c\sqrt{\frac{1}{a^2+b^2}}$$

which can be used to parameterize the problem(s) so x takes the values wanted. We note that c plays the role of an overall scaling parameter and we chose it to be positive, without loss of generality. For simplicity, we recommend taking $a,b > 0$. Note that other cases of a,b are obtained by reflections about the axes. Changing the ratio of a to b changes the slope of the line.

Teaching Point. We support the White-Box/Black-Box Principle first articulated by Bruno Buchberger (1989). Students using CAS should initially include every step of their working until they are secure in their understanding (the White Box), after which they should use the higher level commands provided by the CAS (the Black Box). Maple provides a command that completes the square. Its use should not be permitted until students have mastered the notion and the skills to complete the square "by hand"; either by pen on paper or step by step calculation using the CAS.

Minimize Distance - Geometrically Using Intersecting Lines

This problem is usually presented and solved as a calculus problem. However it can be solved geometrically by finding where two perpendicular lines intersect.

By constructing a line from the origin and perpendicular to the given line, it is obvious (from the right-angled triangle) that where these lines intersect is the point that has the minimum distance from the origin. The perpendicular line passes through the origin, so has equation $y = m\,x$ where m is the slope, see Figure 3. Because it is perpendicular to the given line, its slope must be the negative reciprocal of the slope of the given line. The given line has equation $a\,x + b\,y = c$ in general, and here: $x + 2\,y = 5$.

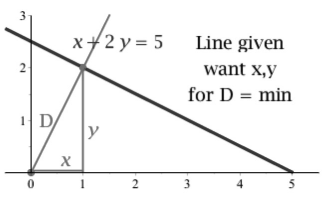

Figure 3. A plot of the given line and the perpendicular line through the origin.

The slope of the given line is -1/2 and so a perpendicular line has slope 2, and if it passes through the origin, its equation is $y = 2\,x$. We solve the two simultaneous equations for the point where these lines intersect by using an obvious substitution: eliminating y in the given line's equation gives $5\,x = 5$.

Thus the x value of the solution is very simply obtained and the y value is obtained, as usual, from the given equation of the line.

Minimize Distance – Geometrically Using Right Angled Triangles

Geometrically, the line of minimum length meets the given line at a right angle. We can find x, y and D by using ratios of corresponding sides of the similar right- angled triangles. To help keep track of the corresponding sides, we mark the same angle with a black dot on the diagram as shown in Figure 4.

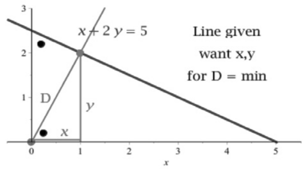

Figure 4. Dots indicate equal angles in similar right-angled triangles.

We choose to use the ratios of opposite side to hypotenuse in the two similar triangles. For the small triangle (in the upper left corner) and the large triangle we have

$D : 5/2 = 5 : 5/2\sqrt{5}$ so $D = 5/\sqrt{5} = \sqrt{5}$

Now we need to find the values of *x* and *y*. Consider the small triangle with hypotenuse of length D, and the previous small triangle (with the hypotenuse along the *y* axis) to obtain

$y : D = D : 5/2$, so $y = 2/5\, D^2 = 2$.

The ratio of sides in our right-angled triangles is $1 : 2 : \sqrt{5}$. So, from the small triangle with hypotenuse of length D : $x = y / 2 = 1$.

Thus, the solution is x = 1, y = 2 for which the minimum $D = \sqrt{5}$

Multiple Representations - Visually

We have used diagrams to represent the problem and also a plot of the objective function, the distance as a function of the one variable. Using a Table in Maple, it is easy to place these representations side by side (and hide the input): the result is the single display of the two plots side by side as in Figure 1.

While this is useful to connect the two visualizations, we can do better by using animation. Students who have worked though our sequence of Maple introduction and many visualizations and animations have become "experts" with animation. Other students might need to have the animation code provided and be asked to edit the code for their parameterization. Our animation takes 21 frames to move the point on the line from the *y* axis to the *x* axis: two frames are shown in Figure 5. The animation shows the diagram with the *x* value and the D value displayed. Running the animation and also manually changing frame by frame reinforces the geometry and the *x*, D values.

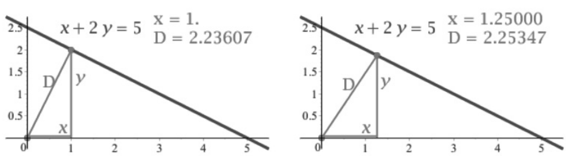

Figure 5. Two frames of an animation of the diagram with the values of x, D displayed.

Animation Using Array

We can create a much more interesting animation that superimposes the dynamic plot of D vs x over the animated diagram. However it gets a bit messy, so it is better to place the 2 representations side by side of each other using the Array command. We wrote a procedure to create one frame (see Figure 6) and then animated this using 21 frames to move the point along the line.

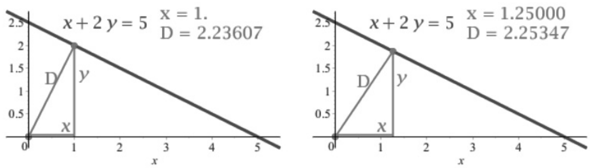

Figure 6. One frame of the geometric diagram side by side with the graph of the objective function. The point on the line is moved from the y axis to the x axis.

Technical and Teaching Note: placing graphics side by side using the Table structure is simple. However animating this presents extra technical challenges. Maple over-rides some details as it automatically creates one separate graphics object (rather than two separate graphics side by side). We recommend that students be provided with the array animation code and be invited to experiment with the animation.

Conclusion

We favour collaborative learning with small groups of students working through activities of the type described in this paper. The problem is best parameterized to foster cooperation without copying. Optimization using quadratic objective functions allows applied problems and problem solving to be introduced early and assists with development of understanding and skills in completing the square. Extension to cubic objective functions can be done without calculus by using a translation and argument in the small approach. For details of this approach and its application to the Open Box problem (see Blyth, 2013). This is a standard calculus problem where square corners are cut from a fixed sheet of material that is then folded to create an open box. The problem asks what are the dimensions such that the maximum volume is obtained?

With parameterization of problems, the marking load increases. One method of handling the marking is to have a master file where the parameters can be adjusted to that of the student work to compare answers. If the student work is submitted as an e-copy, then it can be marked electronically. For example, with a tablet device, Word files can be annotated, pdf files from scanning handwritten work can be annotated with PDF Annotator, and text can be inserted in a Maple file (see Blyth 2012). This "eMarking" is being used increasingly at school and university.

Even better, is the use of CAS and Computer Aided Assessment, CAA. We have written CAA for auto marking within Maple. Our first year undergraduate students love the immediate marking and tend to behave as gamers trying to "clock" a game; students ask for more of these activities (see Blyth & Labovic 2008 and Blyth 2012). Not only does the staff marking load reduce, but the students enjoy the learning experience more! The commercial CAS enabled CAA, MapleTA, uses Maple as its CAS and has an easy-to-use authoring environment: MapleTA can be used as the CAA for "by hand" or CAS based activities.

Although the focus of this paper has been on the "mainstream" use of CAS based activities in the senior school and university, with appropriate modification, they can be used for enrichment activities. For a brief discussion of a successful print, CAS and enrichment in year 9 with a modified senior school Common Assessment Task, CAT, (see Miszkurka, Blyth and Fitz-Gerald 1997). It is interesting to note that the enrichment program was developed to address the fact that a small group of year 9 mathematics students were identified as becoming disinterested with the presentation of standard material. Perhaps universities are not the only institutions where some students are bored by the standard teaching method. The appropriate use of CAS and CAA fosters deep learning and is enjoyable for students and teachers.

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The author thanks Dr Asim Ghous, the Director of ASES, for support with Maple over many years (and MapleTA more recently) and ASES for financial support to prepare this paper and to be a participant at the MAV Conference.

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ONCE UPON A TIME: CHILDREN'S LITERATURE AND MATHEMATICS

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Deakin University

What was your favourite book as a child? Remember the joy of reading it over and over again until the pages were worn and the corners curled. Have you thought about introducing your favourite book into a mathematics classroom? Utilising books in mathematics can engage and benefit every child in your class. Building on children's wonder of literature can enhance their experience in mathematics. Building on children's wonder of mathematics can enhance their experience of literature. In this paper we present the joy and value of employing children's literature in the middle to upper primary mathematics classroom supported with engaging tasks that will have your students noticing maths in every story they read.

The Joy and Value of Literature in the Mathematics Classroom

Employing children's literature in the mathematics classroom can spark the imagination of students in ways that traditional exercises do not (Burns, 2005). Literature makes mathematics come alive (Hoover, 2012) for students who are enchanted and disenchanted with mathematics. Tucker, Boggan, and Harper (2010) suggested children's literature provides a risk-free environment for children to explore and observe mathematical concepts while discovering the link between mathematics and their personal lives. Columba, Kim, and Moe (2005) recommended literature as an effective tool for connecting mathematical ideas. Children's literature reveals mathematics in authentic settings that has meaning to children. Literature also offers fantastical situations in which mathematics is embedded and prompts a curiosity in the story; for example, Counting on Frank by Rod Clement (1990) explores the magnitude of numbers through humourous settings. However, a word of warning is cast by Marston, Muir and Livy (2013) who recommended reviewing the accuracy of the mathematics within a text when they revealed mathematical inaccuracies in Counting on Frank and questioned, "Can you really count on Frank?" (p. 440).

Australian teachers are encouraged to include literature in the mathematics classroom in the *Australian Curriculum: Mathematics* which stated, "Students need to be able to draw on quantitative and spatial information to derive meaning from certain types of texts encountered in the subject of English" (ACARA, 2012). This recommendation for the use of literature is supported by research over the last two decades (see Van den Heuvel-Panhuizen & Elia, 2013; Young-Loveridge, 2004). Anderson, Anderson, and Shapiro (2005) observed the development of mathematical concepts in early years' children through shared reading and discussion of the text. Schiro (1997) suggested the use of literature to promote mathematics instruction, particularly problem-solving and reasoning.

Children's literature offers an introduction into explicit abstract mathematical concepts through books that are purposefully written for the mathematics classroom. Authors who have a strong background in mathematics education, such as Marilyn Burns, have created imaginative mathematics based stories offering students an authentic insight into mathematics (see *Spaghetti and Meatballs for All,* 1997). Equally, there are books not written purposefully for mathematics but that provide affordances for mathematical learning. For example, Roald Dahl's (1964) well-known story of children exploring the delights of a chocolate factory in *Charlie and the Chocolate Factory* offers subtle and non-subtle potential for exploration of probability, mass, volume, ratio, etcetera. Tasks exploring probability, based on this story, are presented later in this paper.

Awareness and Selection of Children's Literature

Whilst non-mathematically specific books offer enormous potential, Whitin and Whitin (2004) claimed identifying mathematics in children's literature was time consuming for teachers and, as a result, may act as an obstacle for immersing students in a literature-based mathematics lesson. Conversely, Perger's (2012) study revealed that Year 7 students could identify opportunities of mathematical learning within samples of children's literature. These students linked literature beyond the mathematics class to alternative areas of the curriculum and life skills. Through the authors' anecdotal observations of teachers and children's engagement with literature, once guided to explore the mathematics in books teachers and students successfully recognise the mathematical potential.

However, while the awareness of mathematics in books is a necessary starting point, guidelines for the selection of books are helpful in determining the usefulness and appropriateness of a text. Marston (2010) offered a framework for selecting picture books for early mathematical development. Marston's framework draws on recent research in early mathematical understanding and the role of visualisation, as well as current pedagogical thinking (Marston et al., 2013). Marston's seven categories for consideration in the framework are: 1) Mathematical content; 2) Curriculum content, policies, and principles; 3) Integration of mathematics content; 4) Mathematical meaning; 5) Mathematical problem solving and reasoning; 6) Affordance for mathematical learning; and 7) Pedagogical implementation. Marston advised that a book does not require all elements to be considered worthwhile for classroom usage. However, the framework does raise the awareness of aspects for consideration and potential mathematical affordances during book selection. Investigation into the use of this framework with chapter books for older children would be worthy of consideration.

A limitation of using children's literature in a mathematics lesson is the potential for teachers to overemphasise the mathematics whilst unintentionally detracting from the literary essence of the text (Shih & Giorgis, 2004). It is important not to "spoil" the book with the mathematics (Hunting, Mousley, & Perry, 2012, p. 60). Nevertheless, "when children's literature and numeracy are connected in an interactive and meaningful way, students will understand the mathematical concepts readily and will sustain the knowledge" (Shartzer, 2008, p. 650). The key is to connect in a "meaningful way". How might this be achieved? The tasks described below provide links between the mathematics and literature that are engaging and raise students' awareness of the mathematics that surrounds them in a meaningful way.

Tasks to Promote Noticing of Mathematics in Books

Designing tasks that offer students the opportunity to notice mathematics in all aspects of their lives is worthwhile as it offers a link with the environment and promotes the valuable daily contributions mathematics makes. The tasks below are linked to authentic and practical mathematical applications whilst stimulating students' noticing of mathematics in literature.

Where is the Mathematics?

A simple but effective approach to exploring the potential for non-mathematics specific books is for children to read a story and list the mathematics they notice and provide an example from the story. A Where is the maths? list is attached to the back of each children's book with the three headings: Mathematics noticed in the story; Example and page number; Who noticed? The list grows as the mathematics the children notice in the text is added. The children compare their observed mathematics to that of their peers; if differences are identified, children often return to the story for a closer examination of the mathematics overlooked. A poster of mathematical content can be available to assist students in identifying mathematics. The poster is created in a brainstorming session with the class listing broadly all the mathematics they can recall. The poster is a work in progress that grows over the year as the students' awareness of the complexities of mathematics develops. The goal of the interrogation of the literature is for children to begin to notice mathematics in aspects of literature. The hope is that they develop Jon Scieszka's (1995) math curse and extend noticing of mathematics to everywhere.

Charlie and the Chocolate Factory

The purpose of this mathematics lesson is to teach the concept of chance for upper primary students through the classic chapter book *Charlie and the Chocolate Factory* (Dahl, 1964). In the chapter titled "The Golden Tickets", Mr Wonka writes a letter stating that only five lucky winners can visit the chocolate factory. The golden tickets are hidden in a Wonka Bar which any person may purchase. Based on the ideas of chance taken from the book, the goal of the lesson is to "describe probabilities using fractions, decimals and percentages (ACMSP144)" (ACARA, 2012,). This engaging, open-ended problem solving task achieves this goal by students examining how small one's chances are of winning competitions based on differing populations, and determining the chance of specific events using mathematical language.

The session commences with the reading of the chapter, next the teacher opens up a discussion about the chance of "winning" the golden ticket in the story. "*What was Charlie's chance of winning the ticket?*" "*How can Charlie improve his chance of winning?*" The teacher sets the task for the students to explore in pairs, "What is your chance of winning a golden ticket if we held the competition in our class?" A string probability line is constructed at the front of the classroom with the numbers 0 and 1 at the ends. A mini "golden ticket" is given to each pair to peg to the probability line to represent their agreed chance of winning the golden ticket in a class competition. The pairs discuss and record in their notebooks their chance of winning represented as a fraction, decimal and percentage. For example, their chance of winning a golden ticket, in a class of 25 students, is 5/25 or 1/5, 0.20 or 20%. The teacher records the different responses of the students in the "Chance Table" on the board with the headings: Number of tickets; Number of People; Fractions; Decimals; and, Percentages. The students are called on by the teacher to justify their selection of a particular representation. Each representation is referred to the equivalent placement on the probability line to emphasise the link between the representations.

The teacher suggests the students look under their chairs to discover if they are heading off on an adventure to the Wonka Factory. The students discover if they are one of the five lucky students to have a shiny golden ticket taped under their chair. Excitement ensues. These students are given a special "Golden Ticket Badge" and become teacher helpers in the Mr Wonka Competition described later in the paper.

Further comparison of the probability of an event occurring is explored through the following questions: "*What is the chance of winning a ticket if Mr Wonka hid 5 tickets in 5 random envelopes, where each student in the school was given an envelope?*" As an enabling prompt, the number of students in the school may be rounded to the closest ten or hundred for a convenient approximation. To build the students' understanding of sample space, number of successes and probability, the following question is posed, "*How many golden tickets would need to be given away by Mr Wonka in the school for you to have the same probability of winning as your class-only competition?*" In small groups, students can brainstorm and investigate different scenarios for the golden ticket raffle to take place, e.g., local shopping centre, the Melbourne Cricket Ground, sporting club. The groups might also alter the number of tickets distributed to compare the probability based on number of successes. A possible scenario is: *What is the chance of winning a golden ticket if Mr Wonka hid 5 tickets under 5 random seats at the MCG?* Students determine where to peg their new scenarios on the probability line and record their estimates in the Chance Table. In a whole class discussion the teacher asks the children to share their findings and what they noticed about the results. Further avenues for discussion might explore the creation of competition for profit, the number of tickets available in raffles and the likelihood of winning a raffle.

The Mr Wonka Competition

Building on the students' experiences with the competitions discussed above, in small groups they create a carnival-type competition and accompanying advertising poster. The task is adapted from an open-ended question designed by Sullivan and Lilburn (2004).

Mr Wonka decides to have a chocolate carnival at Deakin Primary School. There are 1000 students at our school from Prep to Grade 6. He asks our Grade to host several competition booths at the carnival. Create a competition that involves students having a 0.25, 0.5 OR 0.75 chance of winning a prize. Advertise this competition on a poster.

Rules:

- *Competition must involve luck (not strategy)*
- *Choose ONE decimal (0.25, 0.5, 0.75) for your inspiration.*

- *Explain how you will profit from the competition.*
- *Write the competition instructions on the poster; include the price of entering the competition and the prize for winning.*
- *Aim: Make the most money!*

For example, students may make a spinner for a 0.25 chance competition. The circular spinner has 8 slots. If the spinner lands on a colour (2/8) the person gets a prize: If it lands on a white slot (6/8) they do not win.

During a demonstration of the competitions, the groups present their game and poster to the class and explain the game's features. Each child is given 10 gold coins (counters) which buys entry into the competitions of their choice. The students trial the competitions and provide feedback on the quality, engagement and enjoyment. Each group shares the results from the event and the profit they had hoped to gain.

Round and Round and Round and Round

The picture storybook Round and Round and Round and Round by Colin Thompson (2002) is an amusing account of Mrs Golightly's journey of riding a bicycle around the world. Whilst seemingly never lost, Mrs Golightly does not know her various destinations, only knowing the direction she is heading. This lack of awareness regarding her destinations means that Mrs Golightly is not taking the most efficient path to reach all destinations before returning home. Consequently, the story is rich with directional language, emphasising the compass points north, south, east and west. The following tasks build towards students being able to "describe routes using landmarks and directional language (ACMMG113)" (ACARA, 2013).

Before reading the book the students engage in the stimulating activity, "I am north" to raise their awareness of compass points and angles (see Bragg, 2012; Bragg & Skinner, 2011 for the full task description). To summarise, the students are randomly distributed cards which depict a compass point (N, SE, NW) bearing (90°, 45°, 135°) or a compass indicating a cardinal or inter-cardinal point (see Figure 1).

Figure 1. "I am north" compass cards.

The students form a circle, better known as a human compass rose, with the teacher in the centre. The student holding the "N" (for north) reveals their card to the group. The "I am north" student becomes the fixed point on the human compass rose and the remainder of the class reviews their cards and considers their appropriate position. After much discussion, debate, helpful advice and gesturing the students move to their correct position. Arms become compass needles and hands guide the way for peers. The teacher leads a discussion of sharing helpful strategies for finding a compass position. The task is repeated with each student receiving a different compass card and encouraged to trial their peers' strategies.

At the conclusion of the "I am north" task, with the students' heightened awareness of compass points, the teacher reads the story *Round and Round and Round and Round*. The class list the places Mrs Golightly visited on her trip and consider the drawbacks of taking such a journey without knowing the location and only the direction. The teacher prompts the class to consider "*What was the quickest route?*" and "*Why might you not go the quickest route?*".

Students form small groups of 3 or 4 and are provided with a school map highlighting eight locations renamed as places from the book, for example, farm, jungle, outback. The students are instructed to write directions for their peers to go on a school mystery tour which will take them "round and round and round and round" to assorted locations and return home (the classroom) at the end. A requirement of the tour is that it is completed in approximately 20 minutes. This time constraint provides the students with structure when negotiating the locations, directions, and length of each tour leg in metres and time. The mystery tour does not travel to all possible locations and offers a scenic (non-direct) route. There is an emphasis on employing clear directional language and accurate distances. The teacher recaps a prior lesson on how to use a compass (see Bragg & Skinner, 2011, Compass Instructions, pp. 16-17). Once the roughly sketched intended route is determined, with trundle wheel, compass and stopwatch in hand the groups add the necessary directional details such as compass bearings and length in metres to their directions. The students create directions as follows: "*Start at Home (the classroom), Go north 5m, go north-east 17m, Go west 10m, Note your mystery location, Go south-west 6m, go west 12m, Go south-east 6m, Note your mystery location.*" The groups time how long each leg of their journey should take to complete and modify the legs if necessary to fit within the 20 minute time restriction.

In the next session, the teacher distributes photocopies of the directions to another group. Each group takes 5-10 minutes to read the directions and predict where the mystery tour is taking them. The groups record their predictions. Once again with trundle wheel, compass and stopwatch in hand the groups depart on their mystery tour. Each leg of the journey is timed for feedback to the mystery tour designers. A second group trials the map which is also timed. These data from the two groups are analysed by the mystery tour creators and journeys are modified with removal, addition, or alterations to some legs to fit the timing criteria. These maps are offered to a third group to experience. In the next session, the class members predict how long each mystery tour might take if directions to the most efficient, direct route were available. The groups redesign their mystery tour with the most efficient directions and compare the distances and timing of both the first and second tour. The groups discuss the following: "*What strategies might you use to determine the quickest route?*" "*When is it important / not important to go the quickest route?*" At the conclusion of the session the teacher asks the students to complete the following journal prompts to assess and evaluate this task: What is one thing you have learnt today?; What helped you to find that out?; What is one question you still want to ask?; How can you help yourself to find out more about this? (Bragg, Vale, Herbert, Loong, Widjaja, Williams, & Mousley, 2013).

The task could be adapted for younger students by asking each group to develop a route from one location to the next using cardinal points and steps to measure distance. All legs of the journey would be merged to create one Mrs Golightly's mystery tour. While walking to each destination, each group would mark the route travelled and location stops on their copy of a school map.

Conclusion

Teaching mathematics through literature is an engaging and powerful tool, as it is evident that many children are drawn to literature from a very young age (Pound & Lee, 2010). Stories can enhance mathematics, and 'as soon as the mathematical concept is put into a story form it comes alive' (Pound & Lee 2010, p. 66). Similarly, mathematics enriches students' learning experience in literature. The two books discussed above demonstrate how different types of literature, a novel and a picture book, can be effectively used in mathematics. The texts draw upon different mathematical concepts and provide students with the opportunity to explore and experience the mathematics embedded in the text. *Charlie and the Chocolate Factory* is an enchanting novel that engages young readers, who may also dream of winning a golden ticket or competition, to win a prize. The topic of chance is taught in a captivating way to help students not only grasp the mathematical concept, but understand how it applies to the real world. The picture story book Round and Round and Round and Round highlights the directional language of compass points north, south, east and west through Mrs Golightly's adventure around the world. This story provides the basis for an exciting sequence of tasks that uses hands on and practical approaches to teach the concept of directional language. When used purposefully, literature in mathematics is beneficial for students' development of mathematical learning. It is an engaging and exciting tool to ignite students' passion for mathematics at any year level.

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PROMOTING AN AWARENESS OF REASONING IN THE PRIMARY MATHEMATICS CLASSROOM

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Reasoning is essential for developing mathematical proficiency and flexibility. However, often teachers feel unsure how to support reasoning within their classroom activities. The Mathematics Reasoning Research Group (MaRRG) at Deakin University has been exploring ways to promote and assist teachers' and students' awareness of reasoning through tasks and stimulating questions designed to elicit reasoning. In this paper we present a task and a series of thought-provoking prompts developed for primary aged children that were successfully employed to promote reasoning. The task can be adapted and implemented across all primary year levels and differing content areas. It is hoped that this task will further raise the awareness of the importance of reasoning in the mathematical classroom.

The Reason for Reasoning

Reasoning is the foundation of deep mathematical understanding. Its importance is recognised in current curriculum documents in Australia and across the world (Loong, Vale, Bragg, & Herbert, 2013). Reasoning is one of four proficiencies in the *Australia Curriculum: Mathematics* (AC: M) (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2012) where it is defined as "analysing, proving, evaluating, explaining, inferring, justifying and generalising." Reasoning "is the glue that holds everything together, the lodestar that guides learning" (Kilpatrick, Swafford, & Findell, 2001, p. 129). A problem for Australian learners is the general absence of reasoning experiences in the classroom (Stacey, 2003) and the assumption that reasoning is for the talented and senior secondary students (Pletan, Robinson, Berninger, & Abbott, 1995). Stacey (2003) attributes Australia's poor performance in the TIMSS classroom study to "shallow teaching, where students are asked to follow procedures without reasons" (p. 119). Conversely, reasoning can be promoted in the classroom as "students learn to give explanations and justifications when teachers provide tasks that require them to investigate mathematical relationships" (Goos, Stillman, & Vale, 2007, p. 35). The challenge lies in creating investigative tasks that allow for these opportunities.

The Mathematics Reasoning Research Group (MaRRG) at Deakin University is currently undertaking a project aimed at improving teachers' knowledge of reasoning for the creative and critical thinking and engagement of their students. In 2012 and 2013, the MaRRG project conducted two expert-led demonstration lessons in four Victorian and one Canadian primary school in low socio-economic metropolitan and rural areas. A post-lesson discussion held with the teachers observing the session explored what aspects of reasoning they noticed during the demonstration lesson. Within the following two weeks the teachers taught the same lesson or an adaption of it in their classrooms and reported their observations of the children's reasoning, their experience of the task and the structured problem-solving lesson. See Loong, Vale, Bragg, and Herbert (2013) and Herbert, Vale, Bragg, Widjaja, and Loong, (forthcoming) for an analysis of the teachers' experiences in this study. The following task was presented in the first demonstration lesson with middle and upper primary classes, but it can be adapted for all year levels and different mathematical content areas.

What Else Belongs?

"What else belongs?" is an open-ended task inspired by the work of the Canadian educator Marion Small (2011). This task explores mathematical reasoning associated with the idea of "same and different". The learning objectives for the lesson included: students noticing and explaining common properties as similar or dissimilar and justifying this; and students noticing and describing properties of number, such as size, order, composition, place value, multiples, factors, and even or odd.

The lesson commences with a set of three numbers {30, 12, 18} presented to the whole class. The students are asked, "*Think about what you know about each of these numbers. [Pause* *for wait time] I wonder could these numbers belong together. [Pause] What other numbers do you think could belong with these numbers?*" It is important to pause after each question or statement to allow students time to formulate connections between those numbers presented, using their prior experience of the number system. At this stage the teacher does not take verbal responses but rather continues with the stimulating questioning that allows students to simultaneously compare and contrast what is the same and different about these numbers. "*I'm wondering what reasons there might be for these numbers belonging together in this group. [Pause] Why do you think these numbers belong together? [Pause] What is your reason?*" The teacher continues, "*I wonder if there is more than one reason why these numbers belong together. You might be able to think of two or three reasons.*"

After sufficient wait time has passed the students are asked to work with a partner to consider the following questions (see Figure 1).

30 12 18

OUR REASON
These numbers belong together because ….

Other numbers that belong with this group are …

How do you know that all these numbers belong and fit with your reason? Use words, numbers or drawings to explain.

Deakin University Mathematics Reasoning Research Group

Figure 1. What else belongs? worksheet.

During this paired working time, the teacher observes the students' reasoning through these evaluative prompts, "*Can students identify one or more properties (number or numeration facts) that these numbers have in common? Can they give a reason? Can they substantiate their reason – explain how it works for each number? How do they justify their thinking?*" The teacher notes the varying levels of complexity in the students' responses in order to share these later with the class. Anticipated responses for grouping these numbers might be all the numbers are: two-digit numbers; less than …; more than …; even numbers;

multiples of 3 or 6; numbers using digits 0, 1, 2, 3, or 8 (e.g., 82, 13); the tens digit is odd and the ones digit is even (e.g., 52, 74).

The whole class is brought together to share their reasoning. The teacher asks a student, "*You selected [insert number, e.g. 6] Tell us why. Convince us that this reason works for all the numbers [30, 12, 18]*". The teacher uses scaffolding questions to draw out the children's understanding of the nominated reason; that is, why a number is even, multiple of 3, etc. The teacher may ask, "*How do you know these numbers are even? How else can we show that these are even numbers?*" The teacher seeks other members of the class who may have also had "even" as a reason for grouping the numbers and asks them to share alternative numbers they noted that might fit the group. The teacher confirms that these students can support their reasoning by asking, "*How do you know it belongs with this group? How else can we show it belongs?*"

Next, the teacher highlights "difference" by drawing the students' attention to contrasting numbers, "*What numbers would not belong, if this is the reason? Why not?*" The teacher invites one student to respond and convince the class why the number would not belong. For example, the students are expected to explain <u>why</u> a number is "odd" not simply state it is odd.

Students Creating a Set of Numbers That Belong

The next stage of the lesson calls on students to create their own set of numbers and a challenge is posed "*I wonder how many different reasons you and your partner can discover for your group of numbers. Write down the group of three numbers. List your reasons and include at least two other numbers that DO belong and at least two numbers that DON'T belong in the group.*" Children return to working in pairs to select a group of 3 numbers based on a reason that draws these numbers together, and record these on their worksheet (see Figure 2).

Our group of numbers is: _______ _______ _______

Reasons for grouping the numbers	Examples of numbers that DO belong in this group	Examples of numbers that DO NOT belong in this group

Which numbers fit all of your reasons in your list of reasons?________

Figure 2. Worksheet for students to create groups of numbers.

During this period, the teacher's observations may include, "*What properties of numbers are children testing out to make their group of numbers? What reasons are students providing? Do the children need to rethink their selection of numbers to create a greater range of reasons? How many extra numbers are they listing? What variety of numbers is included (e.g., range, negatives, decimal fractions)?*" If students "finish" the task early, they can be given extending prompts such as "*Your numbers are all relatively small: add some more challenging ones*" or "*What about negative numbers or fractions?*" If students are having difficulties with developing the three numbers the teacher may provide them with the following enabling prompts that will start from the reason to group numbers together. For example, "*Can you think of three numbers that are all even?*" or "*Can you think of three numbers that are multiples of 5?*"

An alternative approach is to provide the students with one number which they must include in their group of 3 numbers, e.g., 10. The teacher asks the students "*Tell me anything you know about the number 10*". Based on the students' responses the teacher says, "Okay, can you tell me any other numbers that are [insert a property of the number the student has mentioned, e.g. even number, multiple of 2, 5, 10, etc.]? The students can include these numbers as part of their group of 3.

The teacher draws the class back together to share findings in a final discussion. The teacher uses this final conversation to consolidate understanding of particular number properties, to enable new properties and understandings to emerge, and to emphasise the need to support and explain reasons. The discussion provides further opportunity for children to learn from each other's reasons, examples and explanations. The teacher may identify some pairs of students whose work is displayed and discussed. The first pair should be a group that has one or two reasons for grouping their numbers or has reasons that were commonly identified for the various groups of numbers created by students. The other two pairs of students should have a reason on their list that was not common amongst the class and where there is evidence of creative thinking or the use more advanced or complex reasoning (e.g., multiple of a number greater than 5).

The teacher invites the first selected pair to share their group of numbers (the teacher writes these on the board). The teacher asks the class, "*What might be a reason for grouping these numbers?*" and asks the pair to confirm whether or not these are the reasons. The teacher prompts the class to justify their reason by asking, "*How can we be sure that [insert a number] belongs in this group?" Does anyone else have this reason for grouping their numbers? What numbers did you have? How do you know they belong?*" Through this process the class collectively builds a more complex knowledge of the properties of these numbers and in turn assists the first pair of students in noticing the broader possibilities for their grouping of numbers. The teacher checks in with the pair to determine their current understanding of their group of numbers, "*What do you <u>now</u> know about your group of numbers?*"

The teacher invites the second selected pair to share their group of numbers and any reasons that are different from those offered by the first group. The teacher asks the pair, "*Convince us that this number (one selected from the group) belongs in your group.*" The teacher invites the class to suggest numbers that would belong to the group and explain their reason why the number belongs. Similarly to the first pair, the teacher checks in with second pair about any new knowledge of their group they now have as a result of this sharing process. If there is another pair who has other reasons than those offered in the first two pairs for grouping their numbers then a third pair would be invited to share their group of numbers.

In summarising the discussion the teacher asks the class, "*Why is it important to convince us that a number belongs in the group?*"

Journal Prompts

One way to assess and evaluate aspects of the task that students have become focally aware of during any mathematics session is journal writing. We have found the following journal prompts useful in discovering more about the students' thinking post our whole class and individual discussions.

- What is one thing you have learnt today?
- What helped you to find that out?
- What is one question you still want to ask?
- How can you help yourself to find out more about this?

In response to the question, "*What is one thing you have learnt today?*" children who engaged in the task above noticed the flexibility numbers offer. This is illustrated by the following comment by Tom, "*I learnt that you can reverse numbers, look at numbers in different ways. To combine numbers in different ways with the numbers you already have.*" Ruby in grade 3 was developing her awareness of odd and even through this task. She noticed, "*That if you split an uneven number there will be one left over. That halving thing is easier with a partner.*" Alice observed the complexity of numbers, "*Every number means something by itself or with another number.*" She went on to add, "*Math can be fun and hard.*" This reaffirmed our anecdotal observations that this reasoning focused task was not only challenging but also enjoyable.

Through asking, "What is one question that you still want to ask?" the teacher has an insight into the curiosity of the class, what concepts are still being developed and where to progress with follow up lessons. Eloise highlighted her emerging understanding of even numbers through asking, "*Why is it that if you double a number it always ends up as an even number?*" Other questions demonstrated pondering about the wonder that mathematics offers, for example "*Why [do] numbers go to infinity?*" and "*Why is one not a prime number?*" These are challenging questions to address which offer potential for inquiry-based learning.

Adaptions to the Task

To adapt to the needs of the class and the mathematical content an alternative set of three numbers can be presented. For example, fractions may be a mathematical focus and the following set of numbers may be offered, 5/6, 2/3, 8/9. Exploring the relationship between fractions, decimals and percentages may call for a set of numbers like 0.3, 3/7, 34%. The numbers may be simplified for younger grades to 5 and 10. Shifting to shape, an exploration of a set of rectangles and squares may draw students' focus to the features of four sided figures, however, adding a kite or trapezium to the group will encourage students to notice additional features of quadrilaterals that are not obvious when examining rectangles and squares. See the task *Comparing properties of quadrilaterals* and *Above and below the line* in the AAMT Top Drawer site (http://topdrawer.aamt.edu.au/) for variations of tasks to support reasoning.

Conclusion

"What else belongs?" is one of a range of tasks that can assist students in honing their reasoning capabilities. The Australian Association of Mathematics Teachers (AAMT), with the support of The Mathematical Association of Victoria (MAV), has developed additional curriculum resources for supporting reasoning through the AAMT *Top Drawer* site. The strength of these types of tasks lies not only with the task design but with the thought-provoking prompts utilised to promote reasoning and the teacher's deftness in orchestrating the discussion.

Providing students the space to consider and reflect on the problem is essential for building a respectful environment that allows students to feel comfortable in sharing and conjecturing. Supporting students to think deeply about mathematics is not a trivial task. It requires specific guided questions and prompts like those presented above to facilitate and encourage their reasoning.

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STEP IT UP: USING THE MEASUREMENT CURRICULUM IN THE EARLY YEARS TO CREATE CHALLENGE

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The need to develop a short sequence of lessons on mass measurement with young children led us to consider a different way of planning a unit of work. To create challenge we wrote each lesson so that the cognitive demand of each successive lesson stepped up markedly. We have conceptualized this as a wedge shaped model of planning using the Australian Curriculum: Mathematics. This model of planning has advantages. Planning in this manner enables students to: experience some basic measurement to consolidate earlier learning experiences; take opportunities to extend their conceptual knowledge through experience; be challenged; and experience success. Teachers have opportunities to: focus learning on large "jumps" or "steps" incorporating key ideas; review and refresh children's thinking based on earlier experiences; target the learning goals at the appropriate Year level; extend the children's thinking; and address the range of conceptual development within the classroom with tasks that engage everyone and give opportunities to stretch the children's thinking.

The Background

Many young children could be challenged to measure with greater purpose, understanding and accuracy. We believe that there is room for improvement in the teaching and learning of measurement in the early years of primary school. Based on our experience it seems that various factors contribute to this situation:

- The measurement strand of the mathematics curriculum in the early years is typically not considered central to young children's mathematics learning. Rather, number is usually thought to be the key to children's mathematical futures.
- Measurement is often conceived as a collection of topics to be "covered" by teachers and "fitted into" the planner in what is often considered to be a crowded curriculum.
- Materials and management difficulties are also considered by teachers to be negative factors in the implementation of the measurement stand of the curriculum. Tasks often need adequate measurement equipment and resources that are difficult to access and also require organisation.
- Often teachers do not see the "big ideas" of measurement as they apply to young children. Therefore the focus and clear sense of purpose in the teaching and learning of measurement is lost and the curriculum is implemented as a collection of activities.

Our Work

Our research shows that one topic which receives scant attention is mass measurement. We found that children's learning plateaued in Years 1 and 2 (Cheeseman, McDonough, & Clarke, 2011) and we hypothesised that this was due to limited new experiences being offered to the children in these grades. A survey of 198 teachers of children in the first three years of school revealed that, on average, teachers planned to devote five lessons (approximately five hours) to the teaching and learning of mass concepts over the course of each school year. With this in mind, we set about developing and evaluating a series of five lessons which would engage, challenge and extend children's concepts of mass in Years 1 and 2 (Cheeseman, McDonough, & Ferguson, 2012; McDonough, Cheeseman, & Ferguson, 2012).

We designed lessons that would:

- focus on growth in thinking about measurement concepts,
- involve active measurement by children,
- generate experimentation and investigation by children,

- challenge children to reason mathematically,
- connect to children's lived experiences,
- engage and interest children,
- create opportunities for teacher interaction and questioning,
- provide opportunities for teachers to assess children's thinking, and
- lead to further learning.

We developed five lessons, each of approximately one hour in duration, which could be extended and elaborated by teachers if they wished. The lessons were intended to be conducted in sequence although not necessarily on consecutive days as we were aware of the many constraints of early primary classroom programs.

What is Different Here?

As we have described it so far, not much about the sequence of lessons we designed seems different from the process that many teachers would undertake in developing a measurement topic for their classes. Typically teachers read the intended curriculum (Cuban, 1993; Handal & Herrington, 2003) in this case the Australian Curriculum: Mathematics (Australian Curriculum Assessment and Reporting Authority, 2012) and they look at the Year level(s) at which they teach. Teachers interpret the specific outcome statements then they either think of, or find, tasks that address these intended curriculum outcomes. For example, the curriculum statement for Year 1 for using units of measure states students will "measure and compare the lengths and capacities of pairs of objects using uniform informal units (ACMMG019)" (ACARA, 2012). The teacher would then plan tasks which address this section of the curriculum: possibly considering ways to extend children's thinking into the next specified curriculum statement (see Figure 1) and considering ways of supporting the learning of students who could not yet compare using informal units of measure. The model for this professional thinking is linear, as shown in Figure 1. The focus of teachers' thinking, shown by the darker shaded region, is on the level at which they teach (in this case Year 1) and perhaps a little beyond and a little before as shown by the lightly shaded regions.

Measurement and Geometry: Using units of measurement

Foundation	Year 1	Year 2	Year 3
Use direct and indirect comparisons to decide which is longer, heavier or holds more, and explain reasoning in everyday language (ACMMG006) Measure and compare the lengths and capacities of pairs of objects using uniform informal units (ACMMG019)	Measure and compare the lengths and capacities of pairs of objects using uniform informal units (ACMMG019)	Compare and order several shapes and objects based on length, area, volume and capacity using appropriate uniform informal units (ACMMG037) Compare masses of objects using balance scales (ACMMG038)	Measure, order and compare objects using familiar metric units of length, mass and capacity (ACMMG061)

Figure 1. Diagram of the linear model of curriculum (ACARA, 2012)

This model considers curriculum as a continuum where the Year level determines the section of the curriculum that is "taught". Simultaneously the needs of any children at either end of the range of thinking in the class require attention and planning. The teacher must select tasks which "differentiate" between student needs to ensure that the range of mathematical thinking is addressed. This is a perfectly sensible model as it targets and focuses the intended learning of the students. However, many teachers experience difficultly choosing measurement tasks which address the curriculum statements and address the range of student learning in their classrooms.

We have been investigating the use of measurement tasks which are based on a different model of curriculum and appear to offer potential to both teachers and children (Cheeseman et al., 2012).

In the research project entitled, *Investigating Early Concepts of Mass* (Cheeseman, McDonough, & Ferguson, accepted for publication March 2013) we designed a series of five lessons in mass measurement to be conducted with Years 1 and 2 children (McDonough et al., 2012). As stated earlier, results of a survey of 198 Foundation to Year 2 teachers showed that the average time spent on mass measurement by teachers in the first three years of formal schooling was five hours per year. We considered how, within the constraints of a series of five lessons, all children could be challenged to think deeply about principles of measure and engaged in meaningful mass measurement.

The Australian Curriculum (ACARA, 2012) statements (Fig 1) give a broad sweep from direct and indirect comparisons of measures through the use of informal units to the use of formal units of measure across various attributes. This development of thinking about measures resonated with earlier work in the Early Numeracy Research Project (ENRP) where a learning framework was conceived, of which the part relevant to the research discussed in this paper is shown in Table 1 (Clarke et al., 2002).

Table 1 Early Numeracy Research Project Mass Measurement Framework (Clarke et al., 2002)

0. No apparent awareness of the attribute of mass and its descriptive language.
1. Awareness of the attribute of mass and its descriptive language.
2. Compares, orders, & matches objects by mass.
3. Uses uniform units appropriately, assigning number and unit to the measure.
4. Uses standard units for estimating and measuring mass, with accuracy.
5. Can solve a range of problems involving key concepts of mass.

The framework was originally designed in the light of available research literature to specify large milestones in children's thinking about measurement (Clarke et al., 2002). Using this framework of "growth point" milestones in learning, Clarke et at. examined the research evidence to decide where the learning of children in Years 1 and 2 would typically fit. During the ENRP a one-to-one interview was used to assess children's understandings of mass measurement (Cheeseman et al., 2011), the results of which are presented in Table 2.

Table 2 ENRP students (%) achieving mass growth points in the first three years at school (Clarke et al., 2002)

Summarised "Growth points"	School entry (Mar) (n = 533)	F (Nov) (n = 538)	Year 1 (Nov) (n = 479)	Year 2 (Nov) (n = 256)
0. Not apparent	17	3	1	0
1. Awareness of attribute	15	7	2	0
2. Comparing masses	47	30	17	6
3. Quantifying masses	21	60	69	50
4. Using standard units	0	0	10	38
5. Applying knowledge	0	0	10	38

As Table 2 shows, by the end of their Foundation year most students were able to compare masses, and three-fifths were able to use an informal unit to quantify a mass. By the end of Grade 1, virtually all students were able to compare masses, and four fifths were able to quantify masses and were ready to move towards using standard units. By the end of Grade 2, over 40% were using standard units successfully, and the rest were ready to move towards that goal. These findings provided learning targets for the series of lessons we developed in the *Investigating Early Concepts of Mass* project.

In light of the ENRP data, we designed lessons (tasks) that would begin by establishing a conceptual foundation for the unit of work. That is, the lessons began with a focus on awareness of the attribute and comparison, both big ideas of measurement (McDonough, Cheeseman, & Ferguson, accepted for publication May 2013). The content would then "ramp up" quickly to incorporate the intended curriculum goals and go beyond that traditionally expected of young children. We designed problems that were open and investigative and required children to measure. Each lesson was intended to engage all children and to have the potential to challenge children at their conceptual level of understanding (Fig 2) as illustrated in the following brief descriptions.

Lesson 1: "Tricky" Party Bags

Mathematical focus: Comparing and ordering by hefting and use of balance scales

Materials: Objects in the room; opaque party bags; twist ties for the bags, balance scales if required

Task: With a partner, create and seal three "tricky" party bags. Try to make them so it is hard to tell which is heaviest and which is lightest, that is, so that they are different but close in mass. Together decide which is the heaviest and which is the lightest. Put them in order. Ask another pair of students to compare the masses of your party bags and order them from heaviest to lightest. Where agreement cannot be reached, use a balance scale to check.

Lesson 2: Weighing with Teddies

Mathematical focus: Choosing and using non-standard units to quantify mass

Materials: Plastic "Three Bear" teddies of graduating sizes, balance scales, objects to weigh

Task: With a partner, measure the mass of some things in our room using teddies and the balance scales.

Lesson 3: Post Office Parcels

Mathematical focus: Using standard units to measure mass in grams

Materials: Balance scales, envelopes and parcels, cubic centimetre blocks ("Centicubes") in singles and sticks of ten, Post-it notes

Task: With a partner, use the Centicubes and balance scales to weigh as many envelopes and parcels as you can. Record what you find out.

Lesson 4: Weighing Stations

Mathematical focus: Quantifying mass in non-standard and standard units

Materials: Various materials including fruit and vegetables, packaged foods, balance scales, rice, pasta, non-standard units such as plastic teddy bears and metal washers, standard mass weights (e.g., 5 g, 10 g, 20 g, 1 kg), analogue kitchen scales, and digital scales

Task: Work individually or in small groups. Move around and measure at the workstations.

Lesson 5: Making a Set of Weights

Mathematical focus: Constructing formal units of mass accurately to develop benchmarks for common masses

Materials: play dough, balance scales, standard mass pieces (5 g, 10 g, 20 g), digital scales

Task: Heft and use the balance scales and plastic mass weights to make your own set of play dough weights as close as possible in mass to the actual masses. Then check with a digital scale.

Lesson 1: "Tricky" Party Bags	Lesson 2: Weighing with Teddies	Lesson 3: Post Office Parcels	Lesson 4: Weighing Stations	Lesson 5: Making a Set of Weights
Direct measure Measurement by hefting Comparing, Matching, Ordering and Seriation Measure	**Experiment with informal measures** Use balance scales Equivalent masses	**Measure** Informal to formal units Composite units	**Measure and quantify** Student choice to engage with informal and formal units Measure in varied contexts to reinforce, practice and re-experience measuring mass	**Measure** Use formal units with accuracy to make a set of standard masses
Indirect measure Use of balance scales	**Quantify** Record equivalence as a number of informal units = the mass of an object	**Quantify** Quantifying in tens and ones to approx. 100 Recording in formal units (grams)		**Quantify** Form formal units Check / adjust for accuracy Use digital scales
Cognitive challenge Reasoning about mass Transitive thinking	**Cognitive challenge** Equivalence Interpreting balance scales Reasoning about informal units - smaller units are more accurate Conversion of units	**Cognitive challenge** Using a physical model to create a mental model for 1 gram and 10 gram masses Reunitization Compound units Adding units of mass	**Cognitive challenge** Transfer of learning to new contexts	**Cognitive challenge** Conceive digital display Order masses mentally to an accuracy of 1 gram

Figure 2. A diagram of the sequence of five lessons matched to curriculum outcome statements. A representation of the "wedge" curriculum planning model.

We picture the model for the lessons as a wedge, as is shown in Figure 2, where each lesson represents a substantial cognitive step from the previous lesson. The curriculum content of each lesson is detailed under the lesson title: each column represents one lesson and the sequence of lesson titles 1-5 is read horizontally and the matched curriculum outcomes are aligned horizontally.

Outcomes

Each class had been taught five one-hour lessons over one week, with three of these taught by a researcher who was also a practicing primary school teacher, and two by the classroom teacher (who had observed the researcher teach the lesson earlier that day). Using the relevant section of the ENRP one-to-one interview (Clarke et al., 2002), we assessed 119 Year 1/2 children before and after teaching the lessons described in this paper. The results (as seen in Table 3) showed that the children learned mathematics as a result of participating in the series of five lessons.

Table 3 Difference between teaching experiment children 2nd and 1st interview results

Change from Interview 1 to 2 (in growth points)	Frequency(n=119)	Percent
-1	1	1
0	39	33
1	45	38
2	16	13
3	16	13
4	2	2

One third of the students remained at the same growth point of the mass framework after one week of teaching and learning. This is hardly surprising as in the major research project from which the framework originated (ENRP, Clarke et al., 2002) it was found that a growth point typically took a student about a year to achieve. The growth points were designed to be major milestones in children's mathematical thinking.

About one third (38%) of the teaching experiment students had developed their thinking about mass by one growth point after a week of enriching experiences. And 28% of the students made impressive learning gains of 2, 3 or 4 growth points. The mean was 1.1 growth point gain.

To illustrate the growth possible from the first to the second interview, we can examine the case of Andrew, a Year 1 boy, reported originally in Cheeseman et al. (2012).

> In the first interview, Andrew could heft to compare the mass of items and was able to use the language of heavier and lighter. However, Andrew seemed uncertain about using balance scales and was unable to explain how the scales showed items were heavier or lighter. During the first lesson of the teaching week, Tricky Party Bags [described previously], it was noted that Andrew worked enthusiastically with his partner to discover the difference in mass of two very similar bags. This pair asked to use digital scales so they could "get a number" that would show which was heavier. During the fourth lesson, Andrew was observed persisting for some time to find the mass of one potato using balance scales and a set of mixed weights. After many minutes absorbed in this task, Andrew was able to proudly say that the potato had a mass of 275 grams. In the second interview, three weeks after the teaching experiment had concluded, it was clear that Andrew had made remarkable growth over the course of the teaching week. He was able to use the balance scales to measure with informal units and could also accurately weigh objects on the balance using standard weights giving the answer in numbers and grams. Andrew demonstrated that he was beginning to understand how to use kitchen scales and could explain how the scale worked. Perhaps of most note was that Andrew, a quiet unassuming student, became one of the most enthusiastic and eager participants during the teaching week. (p. 18)

In addition to the learning gains achieved by children, teachers reflected positively on the sequence of lessons and the way they "ramped up so quickly". In fact one teacher asked for "a series of lessons like these for every measurement topic".

The Wedge Planning Model

The intention behind the wedge planning model in measurement is to address curriculum outcomes which are expected of students both before and after the target Year levels. As shown in Figure 3, Years 1 and 2 students would meet curriculum content from Foundation level to Year 3 level in the series of lessons devised for mass measurement.

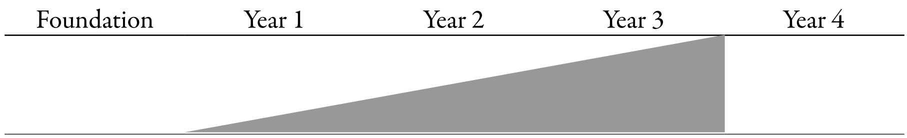

Figure 3. A representation of the spread of the planned lessons across the intended curriculum outcomes.

We see the wedge model as a series of overlapping planning units as shown in Figure 4. In each year there is a reiteration of some of the concepts and skills met the previous year and an extension of them as well.

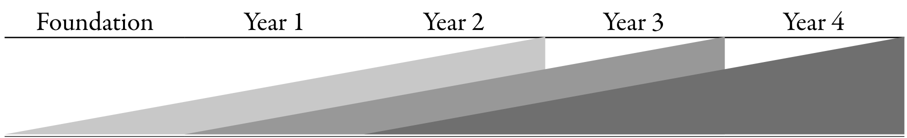

Figure 4. A representation of the overlapping nature of the wedge planning model.

In this way the delivered curriculum spirals upwards but before doing so it folds back on itself each school year to consolidate underpinning concepts in new contexts. Such a planning model offers children a range of experiences which fortify and challenge their measurement concepts and build an awareness of principles of measurement. A range of key ideas for developing understanding of measurement including comparison (direct and indirect), conservation, transitivity, identification of a unit, number assignment, and scale (Clements & Stephan, 2004; Lehrer, 2003; Lehrer, Jaslow & Curtis, 2003; Piaget, Inhelder & Szeminska, 1960; Stephan & Clements, 2003; Wilson & Osborne, 1992) are considered within these lessons. These are important not only for the measurement of mass but transfer across measurement attributes (Wilson & Osborne, 1992) and thus are encountered by children in a range of areas of the measurement curriculum. Other aspects such as quantification and equivalence, important throughout mathematics, are included also.

In using the curriculum model discussed in this paper, we developed whole class lessons in which the same activities are presented to all children in a class. The mathematics in each lesson builds on that in previous lessons. At the same time, each lesson includes a task with a degree of openness that allows children to be both supported and extended in their thinking.

Advantages of Planning Lessons that Address a Wedge of Curriculum

Planning in this manner enables students to:

- experience some basic measurement to consolidate earlier learning experiences;
- take opportunities to extend their conceptual knowledge through experience;
- be challenged; and

- experience success.

Teachers have opportunities to:

- focus learning on large "jumps" or "steps" incorporating key ideas;
- review and refresh children's thinking based on earlier experiences;
- target the learning goals at the appropriate Year level;
- extend the children's thinking; and
- address the range of conceptual development within the classroom with tasks that engage everyone and give opportunities to stretch the children's thinking.

It is also important to stress that the lessons we devised to fit this wedge model of planning had key characteristics: they were investigative and required children to explore a problem; they had some degree of openness; they involved hands-on experiences with increasingly sophisticated mathematical ideas; they built abstraction of key measurement ideas; and built on intuitive understandings and extended prior experiences. In addition teacher behaviours were a feature of the planning. Teachers were required to interact with children in learning conversations; to question and provoke thinking; to elicit reasoning; and to challenge misconceptions. While it could be argued that these characteristics and teacher behaviours are those which are features of sound planning for successful mathematics learning, they were part of the overall planning for the mass measurement lessons described above.

In Conclusion

We offer the wedge model of planning for consideration by teachers as we have found it an effective planning tool for use with the mass measurement curriculum in the Early Years of primary schooling. In this strand of the curriculum at least it seems to cater for the range of learning and to naturally allow teachers to differentiate the curriculum in an inclusive and exciting way.

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THE IMPORTANCE OF LANGUAGE: THE INTERSECTION OF A CROQUEMBOUCHE TIN AND MATHEMATICS

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If we are to teach students to have enquiring minds, wondering and wandering minds, so they embed mathematical ideas and ways of thinking, into their own lived experiences, then it is far more than teaching them the facts. They need to learn how to use mathematical language intuitively, so the embedded ideas, and the nuancing of these ideas can happen naturally. But to start with formal mathematical language is putting the cart before the horse. Starting in the students' realm of their informal everyday language, in which their lived lives are coded, is a much better starting point.

The Croquembouche Story: The Beginning

"Could you drive across and pick up a croquembouche tin tomorrow morning. I've hired one?"

"A what?" says I in somewhat bemused fashion, thinking that yet again I am being sent on a journey to get something of which I have no idea.

"A croquembouche tin, for the cake" said she wondering how come she had ended up with someone so dense.

"A croquembouche tin?" said I again in what I hoped was a gentle enquiring tone.

"Yes. Nic wants to make one himself this time". Seeing the look of continuing bemusement she continued, "The sort that he had for his 21st".

Ah! Now I started to understand. I remembered picking this cake up from the shop a bit before lunchtime on a Sunday and nearly spilling it over the pavement as Simon, who was supposed to be helping and at least holding the car door open just so, gazed after the rather good looking girls as they walked by. Disaster just avoided, we proceeded home to what ended up as quite a memorable party with the cake as the centerpiece.

"Oh you mean the cone thing?"

Her eyes rolled. "Yep that's it. It is actually a croquembouche tin".

And to Mathematics

Now what has this to do with mathematics? Well one connection was the shape; a cone. A second connection perhaps for me was, as a student in this context, not understanding all that much of the conversation, until ...

The point at which I began to understand was when I was given a situational context coached in our everyday informal language to which I could relate (when I had picked up this type of cake some years before). This context was not just the day before, or even during that week; it was five years earlier. But we had a shared history and we were able to recall such joint experiences (well that is different for most shared teacher – student contexts, but nevertheless instructive). The language she used was the way we just chat together.

At this point of beginning to understand what I was required to do, she changed the language of the discussion. She changed it from what was an informative, rich with meaning, and very comfortable language for her, the formal language of cooking, to our everyday language. Interestingly I then changed the language form to what was a comfortable, rich with meaning, and informative language for me, the formal language of mathematics. However what allowed me to make that transition, and hence gain understanding, was her move, good teacher that she is, to moving from the formal language of cooking, to the informal language we both shared to describe a context which we both remembered so I could understand the linkages to her use of the formal cooking language. Note also her looking passed my use of formal mathematical language. I suspect she thought "Well that's is mildly interesting, and useful if he manages to get the correct tin, but really ..." But for me, translating the idea into a language I understood bought with it a nuancing which was important: In my mind's eye I moved away from language and saw the shape of the tin I was confident she wanted me to get.

Using different types of language in teaching is really important, and it is really important in teaching mathematics. We have known language is always present when we are teaching mathematics for a long time, and it is important (Austin & Howson, 1979; Ellerton & Clarkson, 1996). The old, old saying, "I teach mathematics not language", is just not only wrong, it is derelict.

In our teaching of mathematics we need to be very conscious of our own language and our students' language. However, language is too much of an all embracing, monolithic term. We need to become very aware of the different types of languages at our disposal, different aspects of language, and the varieties of ways we can use these to lead our students to deep learning of mathematical ideas. The key question is: how do we teach students to have enquiring minds, wondering and wandering minds, so they embed mathematical ideas and ways of thinking, into their own lived experiences? That will not happen unless we use language well.

Here I will only consider two situations when 'types' of language, among all the other important ideas about language and mathematics, become important in teaching mathematics; one of these situations leads to the other. The first situation, discussed in the next section, parallels what happened with the cake tin. The second situation is taken up in the fourth section.

The Importance of Informal / Everyday Language

Formal language is really important whether it be for cooking or mathematics. For those who know the precision and decisiveness that comes with particular terms of mathematics, and how we use those terms with diagrams and symbols, allows thinking and conversations to occur that brings a host of unspoken understanding embedded in the language. The simplest way to see this is in the vocabulary. Once students know the difference between second and two, they are aware that these words have a common core, but also reference quite different aspects of number. Using the term parallel certainly avoids using a mouthful (lines that never meet), but students become aware of the contexts in which this formal mathematical word often appears (parallelograms, angles on parallel lines, etc.), and hence it can become a cue to the direction in which the conversation is likely to head. There are of course other aspects of formal mathematical language other than vocabulary which students learn such as the preponderance and importance of logical connectives, flagging the underlying logical value embedded in the fabric of mathematics. As well, the sparse use of sentences in mathematical writing, and the ever-present symbols that combined with vocabulary and diagrams to form a powerful way of doing mathematics.

This formal language system can be just as much a barrier for students who do not know, as it is an affordance for those who know. Although the end product may be to learn this formal language, it is not the best place to start. Like many things, the best place to start is where the students are. Virtually all their lived experiences will be encoded in their everyday informal language they use at home, in the playground, visiting relatives, and so on. This informal language is different to the formal language forms expected in school (don't all talk at once, use respectable language when addressing teachers, etc.), and particularly the formal language of mathematics. However there is overlap between a student's informal language, the more structured formal language of school, and indeed the formal language of mathematics. It is best to focus on the intersections or overlaps when introducing students to mathematical language forms.

The mathematics curriculum does give some heed to this approach, but fleetingly. Although there is some urging to use 'everyday language' or 'conventional language' when introducing new concepts to students, there is much more use of terms such as 'language precision', 'difference to everyday language' and 'symbolic context of language'. There needs to be a greater recognition that students encode their lived experiences in their own everyday language. By starting there and moving in a measured manner to the formal language of mathematics, recognising that students use language dynamically switching back and forth as they determine the need, they will gain a deeper understanding of mathematical concepts and realise the connection of these to their own lives, when those connections are present (not all mathematics is 'reality maths').

Such an approach is quite different to the traditional approach of starting with the formal language, even to the extent of starting lessons by writing the formal vocabulary on the board, and then having the students recite them in unison, hoping that the mere memorisation of words would lead to understanding. There are many adults today who can still repeat the simple definition of 'hypotenuse', and some even 'transversal', but cannot remember, if they ever knew, how these definitions named by these terms actually connect with the wider network of mathematical concepts. That is not to say displaying formal terms and their definitions in classrooms, for example on word walls, should not be done. However this should happen after students understand how and when to use such terms, and how they connect with variants and the everyday language that leads to these terms. Maybe the everyday language should be displayed as well, at least during the learning phase.

In taking seriously the notion that students do come to the classroom with much mathematical knowledge, encoded in their everyday language, means they start from a position of building knowledge rather than starting from ground zero. It is always easier and feels much more confortable when you start from what you know than being told you know nothing, or very little, but you have to learn this mathematics stuff, because it is good for you. When you start building your mathematical knowledge from what you already know, it becomes real, and the connections to your own life become that much more obvious.

This assumes that the teacher knows at least something of their students' everyday language and the experiences that are encoded in it. Many teachers clearly continue to update their knowledge of students' informal language as it changes over time, using among other opportunities time spent patrolling the play ground during breaks. However, for a teacher to hear and understand the informal chatter of some student groups can sometimes be a little more difficult whether because of class difference or because of different ethnicity. It is the second of these issues that is now considered.

Lived Student Experiences Encoded in a Language Other Than English

Probably the majority of urban classrooms in Melbourne have at least some bilingual students. There are classrooms where the majority of students are bi or multi lingual, and perhaps more than ten languages may be represented. It is not just urban classrooms as it was say up to five years ago. More and more rural townships also have groups of immigrant families who come from countries where English is not the dominant language.

Nevertheless these bilingual students will encode their lived experiences in their everyday language. Often their everyday language is not English. Looked at in one way this can be perceived, as it traditionally has been, as a hurdle for learning in Australian classrooms, but perceived in another way it opens up affordances for both students and teachers.

For many immigrant students there is no getting around the fact that English is not their dominant home language. Hence for them, much of their everyday lived experiences will be encoded in a language other than English. Traditionally in Australia immigrant families were encouraged to learn English quickly and leave their own culture behind by becoming 'Australian' (code for English speaking families like those that were already here). However clearly that never did happen and most immigrant non-English background families retained their own language and culture but also learnt English and the Australian way of life. That still happens. Sadly recognition of this multicultural emphasis is not always accepted in schooling, particularly in mathematics. But there is a potential affordance for students. Research shows that students who have competence in all of their languages, after other factors such as SES, cognitive development, parental education and such like have been accounted for, have a powerful cognitive edge on other students (Adesope, Lavin, Thompson & Ungerleider, 2010; Cummins, 2000; Slavin & Cheung, 2005), even in mathematics (Barwell, 2009; Moschkovich, 2002; Setati, 2005).

Schools and teachers can do little about factors such as SES and parental education. However they can impact students' language competence. Teachers clearly attend to students' competence in English. Ways also need to be found to ensure, or at least encourage competence in students' first languages, even in the context of mathematics learning. If this happens, then we revert to the earlier argument that if students can access mathematical thinking, or at least their informal mathematical ideas encoded in their informal everyday language, then they have a much firmer base on which to build their complex mathematical thinking and see how this links to their own lives.

Until now we have been thinking of bilingual students. So what then are the affordances for the teacher? One affordance is recognising that the group of students is already in part accessing the mathematical ideas they need to learn, and the students are able to relate these ideas to their own lives: a much easier and indeed exciting context to work within. But it can also be scary not being sure just what students do know when you cannot converse with them in their home language of Vietnamese, Italian, Greek, Farsi, or one of the many other possible languages that might be spoken in your classroom. However, most experienced teachers know that most of their students will be discussing ideas of which they the teacher has little knowledge: how can one really keep track of what all the ideas that a class of more than twenty-five students might be discussing? So a better way forward might be to encourage students to feel free in using their home language if the student thinks that might help them with the mathematical ideas under consideration. Actually research has shown that students will use their home languages in their thinking, and indeed verbal discussions when they have classmates who are co speakers of their home language, whether the teacher is aware of this or not, even when there is no encouragement to do so (Clarkson, 1995). However by encouraging them, the teacher is asking the students to enrich their own languages, and nuance mathematical ideas by thinking about them in multiple languages. But teachers should also ask students to explain their thinking in English, which is what teachers do anyway with English speaking students. By asking the bilingual students to use their English, the teacher is again noting the importance of becoming competent in that language too (Clarkson, 2007).

There is another affordance of multiple languages in the same classroom that is sometimes exploited in other subject areas, but rarely in mathematics. From time to time students are asked to come with history or language information about their home countries, and sometimes these ideas are displayed using in part their home languages (parents love to see that). There is no reason why this cannot also be the case for mathematics. Rather than being confusing, examining together why different algorithms, used in different countries, work for the four basic operations can be a very enriching exercise and deepen understanding of underlying mathematical notions. By understanding that different mathematical symbolisation used in different countries does not necessarily lead to mystifying mathematics, rather it can lead to the fundamental idea of the frequent use of convention, and what that means in the whole mathematical structure. It can even lead to notions that different cultures have not always relied on western mathematics, and there are still other mathematical systems alive and well and in use in other parts of the world (Barton, 2010). It is a good thing to know that our way of life is not all pervading in this world of ours.

The Croquembouche Story: The End

And so, back to the croquembouche, and my own learning trajectory: Well it turns out that the tin was not a cone but really a truncated cone: the vertex is sliced off. When I mentioned this after getting home next morning I was greeted with rolled eyes. I assumed being a tin, it would be filled with the profiteroles, which themselves were filled with different types of icing. Well no. This recipe called for using the truncated cone as an inside structure, and building up the cake with the icing filled profiteroles on its outside. BUT ..., whatever: My knowledge of formal cooking language increased, and more importantly the resulting croquembouche was wonderful to eat, and a fitting center piece for yet another terrific party.

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THE REASONING PROFICIENCY

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If students are to reason mathematically, they need to be engaged in mathematically rich, investigative tasks that allow them to explain their thinking, justify the strategies they use and the conclusions they reach, and adapt the known to the unknown. The importance of contextualised learning should be highlighted so students may be encouraged to transfer their learning from one context to another, explain their choices within a context, and compare and contrast related ideas. The reasoning proficiency naturally interrelates with the understanding, problem solving and fluency proficiencies.

The Australian Curriculum: Mathematics (ACM) (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2013) is not a syllabus. It does more than tell teachers what to teach, it describes how the learning environment needs to be constructed. Through the proficiencies it articulates the constructivist basis that is expected, and gives direction on how this can be achieved. The ACM is a document about the holistic development of numeracy capabilities rather than just a set of essential skills and knowledge (ACARA, 2013).

In 2010, ACARA proposed that the ACM should be constructed around three content areas; Measurement and Geometry, Number and Algebra, and Statistics and Probability. Sullivan (2012) calls these the 'nouns' of the curriculum. ACARA further proposed there should be four proficiency strands; Understanding, Problem Solving, Fluency and Reasoning, and that these proficiencies should be enacted when learning mathematics not just when applying it (Askew, 2012b). Sullivan (2012) refers to the proficiencies as being the 'verbs' of the curriculum. The importance of these 'verbs' is echoed by Burns (2012), who states that the Standards for Mathematical Practice, the US equivalent of the proficiency strands, should be at the forefront of mathematics teaching and learning.

The proficiencies were adapted from the work of Kilpatrick, Swafford and Findell (National Research Council, 2001) who articulated five strands:

- conceptual understanding—comprehension of mathematical concepts, operations, and relations
- procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- strategic competence—ability to formulate, represent, and solve mathematical problems
- adaptive reasoning—capacity for logical thought, reflection, explanation, and justification
- productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy. (p.5)

They saw the strands as being interdependent and interwoven and in fact illustrated it as such. This 'weave' was considered essential.

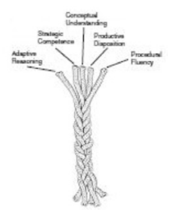

Figure 1. Intertwined strands of proficiency (National Research Council, 2001, p. 5)

Prior to the ACM, states such as Western Australia employed their own curricula and included something referred to as 'Working Mathematically' (Curriculum Council, 1998). Working Mathematically had the same essential elements as the proficiencies, but laboured under the issue of being seen in the same vein as the content strands. In fact rather than being something which was incorporated into all of the other strands as intended, it was often seen as another lesson and timetabled as 'problem solving'. Although this was not the intent, for many teachers it became the practice.

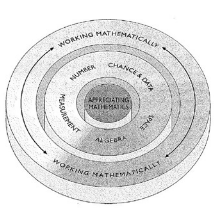

Figure 2. Working Mathematically (Curriculum Council, 1998, p. 182)

The proficiencies are positioned and emphasised to try to avoid the difficulties encountered in trying to get the Working Mathematically as a natural part of teachers' views when constructing every mathematics lesson. They are intended to be an integral part of the curriculum rather than added extras, as they form the foundation on which to build the content areas. This paper will investigate the Reasoning Proficiency in particular, although it should be noted that all four are naturally intertwined with each other with each one reliant on the others to become evident.

Reasoning

Lannin, Ellis and Elliot (2010) explain that mathematical reasoning is an evolving process. The development of a classroom culture where the quality of the mathematical justification that is provided determines the correctness of a response and students are encouraged to take risks by sharing their reasoning (Clarke, Roche & van der Schans, 2012). A climate where students feel safe to share correct and incorrect ideas allows students to develop an "increasingly sophisticated capacity for logical thought and actions" (ACARA, 2013, p. 6).

Students are reasoning mathematically when they explain their thinking, when they deduce and justify strategies used and conclusions reached, when they adapt the known to the unknown, when they transfer learning from one context to another, when they prove that something is true or false and when they compare and contrast related ideas and explain their choices (ACARA, 2013, p. 6)

The teaching and learning of mathematics is about interactions between the students, the teacher and the mathematics (National Research Council, 2001). In order to facilitate these interactions, explicit links between written and spoken mathematics, problem settings and students' solution methods should be made. To this end students and teachers should be encouraged to make conjectures, analyse, prove, investigate, explain, infer, justify, and develop and evaluate arguments (ACARA, 2013; Lannin et al., 2011).

Mathematical Discourse

Small and whole-group settings allow students to share and debate the reasons why they believe a statement to be true and hold all students accountable for understanding (Clarke et al., 2012). Askew (2012a) talks about private talk and public conversations, where the former allows students to engage with the mathematics being discussed and share their thinking in a safe setting. The small group setting provides students with an opportunity to rehearse and refine the ideas that they might share more widely. The public conversation provides a platform for students and teachers to share and build on ideas, emphasise and model mathematical reasoning and problem solving (Kilpatrick et al., 2001).

In order for this to be achieved, mathematical discourse should be well planned to develop students' understanding of key ideas. It should not be confined just to answers, but should include discussion about connections to other problems, multiple representations and solution strategies, justification, argumentation and generalisation (Kilpatrick et al., 2001). It should not be restricted to 'show and tell' or only 'checking for understanding', although the latter may be one of the goals of the conversation.

Rich Investigative Mathematical Tasks

The two activities that assist to improve mathematical reasoning are the engagement in mathematical thinking by solving problems and investigating mathematical situations, and reflecting on the experience (Siemon et al., 2011). Mathematically rich investigative tasks provide the opportunity to transform school mathematics from a collection of rules without meaning to a vibrant, connected subject where there is the opportunity to understand, explore and reason about mathematical concepts. Kilpatrick et al., (2001) suggest that mathematical tasks are central to students' learning, as they shape the manner in which students view the mathematics as well as providing the opportunity for students to learn the mathematics. They go on to point out that the cognitive demand of tasks can vary significantly.

The best rich investigative tasks allow students to work like mathematicians, and see others working like mathematicians, by allowing them to get started and explore, while still providing opportunities for challenge, cognitive demand and extension. They cater for student diversity as a result of their openness, encourage a variety of learning styles and expect students to explain their thinking. Within meaningful or intriguing contexts they develop thinking, reasoning and communication skills while seeking deep understandings. They highlight the interdisciplinary connections both within and outside of mathematics and use information communication technologies effectively (Day, 2012; Lovitt & Clarke, 2011).

Kilpatrick et al. (2001) contend that problem solving should be where all of the mathematical proficiencies come together, to provide an avenue for students to weave all of the proficiencies together and allow teachers to assess student performance on all of the proficiency strands. The use of rich investigative tasks provides an opportunity for mathematical teaching and learning to become more concerned with thinking, reasoning, communication and justification rather than facts, skills and rules without meaning. This is the reason for the importance attached to the mathematical proficiencies within the Australian Curriculum: Mathematics.

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USING AN IPAD IN A MATHEMATICS CLASS

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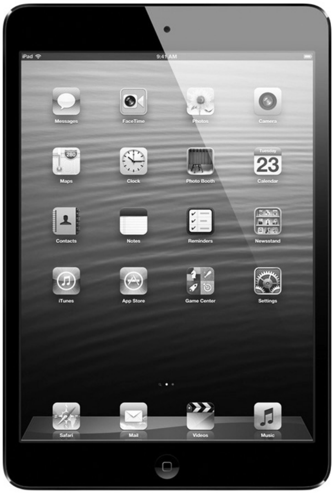
A number of schools have introduced some form of mobile technology into their classes, either Notebooks or the more recent Apple iPads and have then started to find uses for them in the classes. The iPads give students access to a number of problem solving applications, some open ended, that can be done in class and allow teachers to add another approach to their range of teaching techniques.

iPads in Education

What are iPads?

iPads are basically small personal computers developed by Apple computers. As with most Apple products they work with Apple approved products only. There are fortunately many software Applications called "apps" that can be downloaded using the supplied iTunes program with many apps being free. They are fortunately quite easy to use and students should be able to use them intuitively if they have already had some computer access beforehand. The iPads use the standard interface that is common to most computers and a touch sensitive screen and there are many stylus type pens as well. It is well worth the money to buy a protective case, a number of which come with a portable keyboard.



The first and most obvious use of these technologies is as a gate to the Internet through Wi-Fi access and a program such as Internet Explorer or Safari. This is also one of the main concerns that a school needs to consider when developing an implementation plan so that Internet access is appropriate for the class program. Students will often go onto Facebook or similar sites when they have the opportunity and they can spend a lot of time online less productively than would be desired. There are a number of suggested implementation plans on education websites.

Text Books

The recently published textbooks from most major suppliers come with digital versions, either on a disk or via website code redemption, normally as PDF files. These files can be easily uploaded to the iPad using the iTunes "add file to library" or "add folder to library". The students then do not need to cart their heavy textbooks around and they will have a textbook with them as they will make sure that they have their iPads at all times.

At recent book publisher presentations, the various major publishers stated that they are working to make their material more iPad friendly and some have produced specific apps that use the code that comes with their texts. It is also handy as a teacher to project the textbook questions using interactive whiteboards or data projectors although most of the teacher editions come with answers.

Teacher Utilities

There are a number of applications that are available for teacher use and the following are some that have been tested in classes.

Socrative

"Socrative is a smart student response system that empowers teachers to engage their classrooms through a series of educational exercises and games via smart phones, laptops, and tablets." (www.socrative.com) This is an example of "clicker" technology where students give responses to questions, normally multiple choice, that are then collated and displayed automatically when connected to a data projector. This is, perhaps surprisingly, very engaging for the students as they can present their answers anonymously and the class can discuss why some incorrect answers may occur. The discussion of how errors occur can be just as beneficial as going through the correct processes.

This particular software, which is FREE, is easy to use and does allow a number of question types so that you can have the students do the questions and collect their answers automatically – good for the environmentally concerned of us. It does take a short while to create and upload the tests but they are then reusable and can be shared easily.

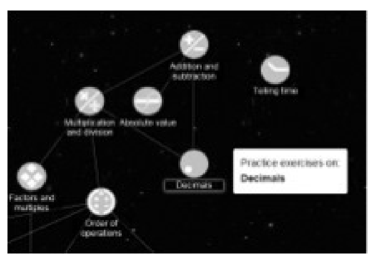
Other Interesting Educational Apps

There are a number of websites that are available through iPad apps which can be used for classes and also for teacher professional development.

Ted.com

Ted.com is a collection of videos covering a wide range of topics. The app allows you to download and then watch a video at a later time. The videos cover talks from leading experts giving very engaging presentations including topics as broad as "Why is x the unknown?" (http://www.ted.com/talks/lang/en/terry_moore_why_is_x_the_unknown/) and there is a very interesting explanation for this. The talks are likely to be advanced for many students although they will also be of interest to teachers.

Khan academy



The Khan Academy, www.khanacademy.org, is an iPad app that allows access, with an internet connection, to the Khan Academy collection of over 3000 videos that cover a massive number of topics. These videos can be used by students to go over a topic or to review the concepts at a later time. The videos are detailed and free. They are accessible by a search menu or through a graphical interface.

Dropbox

Dropbox is a web 'cloud' application that can be installed on iPads or laptops. It creates a folder that you can invite selected students to 'Join" so that they have internet access to this folder and can copy material from it. It is useful for 'dropping' in sets of notes, old tests, solutions, homework materials and saves a lot of paper as well as photocopy costs. You could drop in a test at the start of a lesson and set it as read only and later add the solutions as well.

Finding Apps for iPads

There are a number of iPad apps available, literally thousands, many of which are free. There are a number of websites that make recommendations relating to educational apps although a simple Google search, such as "educational iPad apps mathematics" will find plenty of examples and they will normally connect straight to iTunes and can be downloaded to the iPad directly. Some starting places for mathematics apps are:

http://appsineducation.blogspot.com.au/p/maths-ipad-apps.html

http://www.mathsinsider.com/16-cool-ipad-math-apps-that-your-child-might-actually-love/

http://mathxtc.com/MacMaths/iPadMaths/iPadMaths.html

Using the iPad Camera

Students often use the camera to record whiteboard notes, either so they don't have to write them down or to pass onto friends. There are also some apps that will record your screen so that you can keep a video copy of any notes you might use when it is used as a data projector. The screen resolution and stylus pen accuracy are not very good at the moment so the pen can be difficult to use unless you have good writing skills.

Student Apps

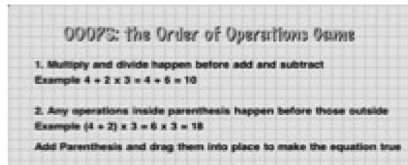
There are numerous apps available for students to use in their mathematics classes. Although the cost of most apps is quite low, often $0.99, the total cost can build up. Most apps are described well in their iTunes pages although a few that are 'free' are only free for the beginning level and more levels or chapters can be expensive. On rare occasions some of the apps are not as described and one that showed various steps of calculus operations was actually doing the calculations incorrectly. It is best to search for apps related to a topic such as 'probability' and see what is available. The following apps are examples related to specific topics and include some problem solving examples.

Minds of Mathematics

This is an extensive timeline covering the major discoveries in mathematics with articles and detailed commentaries. The app is however large and should be downloaded on a fast connection.

Ooops

This is a simple app that asks the students to add brackets at the correct location to make equations true.



http://itunes.apple.com/us/app/ooops/id467564672?mt=8&affId=1449142

It seems very simple at the start but does become quite complex as it gets to the higher levels.

Learning Programs

There are a number of apps that will solve quadratic equations, such as "Factor". There are games like 'Bubbles" that require students to use their skills with the four operations in order to complete various levels. The old 15 slider puzzle is also available free.

Puzzles for Students

The available apps include a number of Tangram based puzzles. Other free pattern puzzles are "MindPuzzle" and "Water Pipes". Apart from a number of actual Sudoku problems there are also variations including KenKen which is an extension to Sudoku with mathematical questions – very addictive! Other examples include the "8 Queens" problem, the "Knights Tour", mazes, chess games, code puzzles and dice based games such as Yahtzee.

Now go to Apple.com, download iTunes and start exploring!

A THERAPEUTIC DISPOSITION

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Parkville College

The literature about Numeracy teaching abounds with curriculum and assessment advice. Students' dispositions related to Numeracy have been explored in-depth. With the focus on the sizeable effect size of the classroom teacher, it is rarely acknowledged how all of those elements are largely contingent on the dispositions of the teacher. This article investigates the dispositions of a highly effective Numeracy teacher, and how those values and principles can engage even the most disenfranchised students.

Disposition

One of the recurring themes in Numeracy literature is the reference to a student's 'disposition' required to engage in learning (Ginsburg, Manly & Schmitt, 2006; Kilpatrick, Swafford & Findell, 2001; Sullivan, 2011). It is acknowledged that a student will only embark on an educational journey if they are "emotionally able and willing" and feel relatively confident "dealing with possible confusion, frustration or ambiguity as it arises" (Ginsburg, Manly & Schmitt, 2006, p. 21). The disposition of the learner is justifiably included as a key component of the Numeracy learning experience. The answer to addressing this crucial facet of the Numeracy learner's experience is often attributed to the structure or content of the class, and infrequently written about in relation to the disposition of the teacher. If the very nature of learning and teaching rests upon the principle that "the learning environment is supportive and productive" (DEECD, 2012), then it follows that the thoughts, behaviours and emotions of the teacher will have a critical impact on the learner's environment, and therefore their learned dispositions. The question arises, what should be the guiding values of a teacher's disposition in order to teach positive dispositions and engage all students in significant Numeracy learning?

Context

In 1957, influential psychologist Carl Rogers wrote his seminal paper that described "the necessary and sufficient conditions of therapeutic personality change" (Rogers, 2007, p. 240). Just over a decade later he had adapted those theories into a comprehensive vision of education. It is largely Rogers' theories, informed by contemporary trauma theory and motivational interviewing methods (Downey, 2007; Miller & Rose, 2009), that have informed the training and dispositions of teachers at Parkville College, Victoria's newest school located within the Parkville Youth Justice Precinct.

Parkville College is a school with arguably Victoria's most disengaged students. By its very nature as a youth custodial setting, the students would never aspire to be there, a unique challenge for a school. Yet students are promised and then involved in meaningful Numeracy learning experiences. I will describe the underpinning theories driving the dispositions of the teachers at Parkville College, in the hope that it will serve as a model to engage disengaged and disenfranchised students, and deepen the relationship between teachers and all their students.

The Necessary and Sufficient Conditions

As teachers, we are always inviting our students to change and grow. We prepare lessons that we think are imperative to their future wellbeing. We manipulate a multitude of variables in order to provide the best environment in which to learn those lessons. We ask that students listen to what we have to offer, try it for themselves and potentially integrate our offerings into their lives. This process renders both the teacher and the student vulnerable. There are issues of trust, power and authority intrinsically weaved into those interactions. In order to create the safest and most supportive environment, Rogers posited several conditions. He wrote about them in terms of 'constructive personality change', and was adamant that they referred not just to psychotherapy, that they would "hold in any situation" (Rogers, 2007, p. 245).

The conditions the teacher must integrate into their disposition in order for a productive teacher/student relationship to occur can be paraphrased as follows:

1. The teacher experiences and expresses "unconditional positive regard" (Rogers, p. 241). Which is tied into the notion that
2. the teacher has a rich empathy for the student. And that
3. the teacher is "genuine" (Rogers, p. 242) in the relationship with the student.

Unconditional Positive Regard

When we arrive to a class with the idea that we will teach division, we very often neglect to consider the inner worlds of each of our students. Perhaps they didn't sleep the night before. Maybe they haven't had enough to eat. There is the distinct possibility that one of them has a vexing and real numeracy problem to deal with that has very little to do with division.

The notion of 'unconditional positive regard' is the feeling of "warm acceptance" for our students (Rogers, 2007, p. 243). Even if a student expresses resistance to learning division, it is incumbent on us to care for that student and their needs. Their openness to engaging with division at that particular moment in time is not the factor upon which our regard for that student hinges. In fact, there should be no precondition for a positive regard for each and every one of our students if we are to engage our students in positive growth.

This type of teacher disposition manifests itself in many ways. A teacher who genuinely feels unconditional positive regard for their students will experience a student's reluctance to engage not as a personal slight, but as a sign that the student is in need of something. The regard the teacher holds for their student is based on assuming the best of the student. Some of the primary questions a teacher with unconditional positive regard might ask themselves when confronted with a reluctant learner are, did I adequately explain why I thought the lesson would benefit the student? Or perhaps, did I misjudge the needs of my students at this point in time?

Empathy and Genuineness

Division can be subjectively thought of as useless and boring. Whether or not we feel that sentiment ourselves, it is important to acknowledge that it exists and will almost certainly be the lived experience of many of our students. An empathic understanding of our students allows us to prime ourselves to respond effectively to the inevitable reticence to participate in a Numeracy class.

Genuineness ensures we are building strong *and* honest relationships with our students. As much as we might empathise with a student who hates division, we are required to teach it at some point and the student will benefit from learning it. The idea that the student refusing to participate in a class about division is hindering our efforts to educate them can be very annoying. After all, we are acting in their best interests and they are often half asleep with the pen we lent them hanging out of their mouth. It is at this point that 'Calmer Classrooms' (Downey, 2007) recommends managing our reactions and offering structured choices. I will describe the elements using an example.

A Therapeutic Approach

When embodying the full suite of dispositions already mentioned above, a teacher might approach the reluctant learner as follows.

> *"I really appreciate that you came to class today, and I can see that you don't seem too excited about division right now, I'd love to get you started by showing you one of the first few questions and see if that helps?"*

It is worth unpacking that statement to see its composite parts.

"I really appreciate that you came to class today..." This shows the student that you have unconditional positive regard for them. The student isn't participating, but there are some choices they have made that we do appreciate, such as showing up. Our regard for the student doesn't rely on their choices, so it is worth voicing some positive reinforcement when they make good ones, no matter how small (Downey, p. 22).

"...and I can see that you don't seem too excited about division right now..." By noticing that the student isn't enjoying themself, we are subtly expressing empathy. By non-judgmentally labeling how they are presenting themselves, we are reflecting how they might be perceived. They may not have known that it was so obvious they look bored and might learn an important lesson from finding that out (Downey, p. 24). Whatever the outcome, it certainly demonstrates that you have thought about them and their feelings.

"I'd love to get you started by showing you one of the first few questions and see if that helps?" The final phrase has a lot within it. It expresses genuineness and offers so much. I would genuinely love to get the student started. I think offering to do an example offers a unity and camaraderie in the educational process. It also offers a quite strongly directed choice (Downey, p. 21). The choice is limited to accepting help, or not, but the nuanced phrasing certainly makes it more difficult for the student to reject assistance, and therefore helps them to accept it.

An important part of the whole approach I have been exploring is that the student in the interaction may choose to opt out of the process. They may, politely or not, refuse assistance and not participate. Managing our reactions and remaining calm is tied up in understanding that the stronger the bond we have with our students, the more likely they are to engage (Downey, 20). They may never appreciate the "elegance" (Sullivan, 2011, p. 5) of mathematical thinking, but without us acting in the best interests of our relationship with them, and therefore their education, they may also never appreciate its utility. At the forefront of our thoughts when dealing with difficult students should always be the ultimate aim of providing them with the highest quality education. Paradoxically, with our students who often resist, this might mean respecting their choice to not participate, whilst always guiding them gently back towards engagement.

Guiding Towards Engagement

Prior to receiving adequate training, I often felt that my one-to-one 'motivational' conversations with disengaged students would reinforce not much more than the idea that I was nice. The student would continue acting the way they had before our conversation, with a brief improvement out of respect for the fact that I had remained nice. The therapeutic approach guiding a teacher's dispositions that I've described can assist in forming and maintaining a relationship with a student, but how can that relationship be capitalized upon in the best interests of the student and their learning?

Motivational Interviewing

Motivational interviewing (MI) is a "collaborative conversation to strengthen a person's own motivation for and commitment to change" (Wilhelm, 2011). There are three principles of MI that are not already encompassed in the therapeutic approach already described (Wilhelm). They are,

4. Support self-efficacy.
5. Develop discrepancy.
6. Roll with resistance.

Many teachers will already be familiar with the idea of supporting self-efficacy. Celebrating a student's past successes to support the belief that the student has it within themself to succeed is often second nature to a teacher. Developing discrepancy is also quite central to common teaching practice. The ability to contrast a student's current behaviours with their values is an effective tool that many teachers employ. Rolling with resistance is a method that yields effective results (Miller & Rose, 2009) and in many situations, is the antithesis of how many teachers are taught and expected to act.

A student will rarely change because we tell them to, they will change because they want to. It is for that reason that a skilled MI practitioner understands that their disposition is crucial. Often, resistance on the part of the student means that they are not ready to change, and pushing ahead is unlikely to change the student's behaviour or attitude in the short or long term (Miller & Rose, 2009, p. 9). If a student refuses to engage with a task, a teacher who is rolling with resistance may offer them an alternative task, or alternative conditions under which to complete the task. If they are flexible and genuine enough, and if their school leadership supports them, they may even ask, "is there something else that you would rather be doing?"

The disposition of a teacher utilising these strategies to their fullest extent is calm and in control. They feel empowered and empathic. They feel a real sense of unity with their students and will, in my experience, find themselves enjoying their time in class.

A Caveat

Teachers are constantly building meaningful relationships with a large number of students. After the school day is over, we go home and interact with a complex set of friends, family, partners and whoever comes into our lives. Into each one of these "interpersonal encounters we bring our buried history of wishes, fears, and psychic traumas" (Kahn, 2002, p. 200). Many teachers have had the experience of being addressed as "Dad" or "Mum". The student then turns a bright shade of red to the familiar chorus of accusatory laughter. It was Sigmund Freud who observed that patients "could see the therapist as the critical father or the nurturing mother or a competitive sibling" (Kahn, p. 184). Patients' transferring their feelings onto their therapist was termed transference and similar processes are happening in our classrooms. Although a student calling a teacher "Dad" is not necessarily transference, it certainly illuminates something occurring within the thoughts of the student.

The caveat that is pertinent to this entire discussion is the notion of countertransference, defined as, "all the feelings, thoughts, and perceptions the therapist has about the client" (Kahn, p. 198). In teaching terms, what the teacher feels, thinks and perceives about their students. By its very nature, countertransference is essential for empathy (Kahn). Countertransference is a necessary process for us to bring our students into our emotional lives. What we need to be especially wary of is the "very common need to replay old traumatic situations or old traumatic relationships" (Kahn., p. 183). As teachers, we need to be careful to look after our mental health so that we are not participating in an unfortunate emotional pantomime with our students. Students, and particularly students who find and push our emotional buttons, can unwittingly be on the receiving end of emotions and reactions intended for someone else. An emotionally aware and secure Numeracy teacher will be able to teach disengaged students better than someone who is unaware of how their emotions impact on them.

Conclusions

There is enough literature in the Numeracy canon to find answers to questions of curriculum and assessment. What I have tried to address are issues relating to how a teacher can prepare themselves in terms of their dispositions and interactions with students. Rogers' (2007) notions of unconditional positive regard, empathy and genuineness provide a solid basis for productive interactions with students. Motivational Interviewing (Wilhelm, 2011) provides the structures and methods, such as rolling with resistance, to build on positive relationships towards real change. Calmer Classrooms (Downey, 2007) is particularly helpful when trying to understand the inner world of a traumatised student, and instructive in providing certain methods to create a safe and productive classroom.

A therapeutic approach to teaching is beneficial for both teachers and students. In my experience it creates a more joyous and productive classroom with all students, regardless of their ethnicity, mental state or attitudes to school. Students who feel cared for, respected and heard will inevitably be more productive. In my experience, teachers who feel they have a deep and research-based understanding of the dispositions and actions that can create that type of environment feel empowered, and happy.

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FROM NUMBER SENSE TO FLUENCY USING PLAYING CARDS

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> *"Almost all creativity involves purposeful play." ~Abraham Maslow, 1908-1970*
> *"Do not…keep children to their studies by compulsion but by play." ~Plato, 427-347 BCE*

That there is a prevailing negative attitude towards mathematics amongst many in the population is well documented. For some, their negative attitude towards the subject of mathematics starts before they even enter kindergarten (Arnold, Fisher, Doctoroff & Dobbs, 2002). Factors such as the child's educational context at home (Scarpello, 2007), being socialised to dislike mathematics because of gender (Geist & King, 2008) or ethnic background (Scarpello, 2007), teacher influences (Stuart, 2000) or failure in the subject can all contribute.

Whilst accepting that these all contribute to negative attitudes it is incumbent on teachers to provide mathematics that at least gives the opportunity for these factors to be ameliorated if not completely dispelled. The work of Hattie (2003) shows that apart from what the child "brings to the table" the teacher and the quality of what the teacher brings to the classroom has by far the biggest effect on the learning of students. One tool, for providing an alternative pathway to robust mathematical learning is games.

Games have many advantages, they generate enthusiasm (Bragg, 2006), they add variety to a mathematics classroom, they encourage mathematical discussion (Ernest, 1986), students are actively involved (Pinter, 2011), they require co-operative learning (Ernest, 1986) and they provide visual representations of problems through manipulative operations (Pinter, 2011). These are all elements that speak of engagement and a game that engages has the capacity to embed learning (Parsons, 2008) with opportunities for reinforcement and practice of skills, acquisition and development of concepts as well as the development of problem solving strategies (Ernest, 1986).

In this paper we will be concentrating on card games in mathematics and in particular card games that fall into Parsons' (2008) broad category of "luck with skill" games (p. 209). Parsons categorises games into three groups, luck, a mixture of luck with skill and skill. He claims that students quickly disengage with luck games and games of pure skill can result in an imbalance which one person may find disengaging and that games with a combination of luck and skill are best (2009). The following games have this combination. The focus will be on the development of number sense and mental strategies with the aim of improving computational fluency.

Make 10

This game is eponymous, so the aim of this game is to make totals to ten, by using as many cards as possible. In this article we'll be exploring using only the operation of addition but this game does lend itself to the use of other operations once the understanding has been developed.

Ten of course is important in our number system and partitioning firstly to ten and then through ten, is essential in developing a fluency which will ultimately be beneficial in the understanding of place value.

All of the picture cards are removed from a deck of cards and the deck is thoroughly shuffled. Ten cards are then placed face-up on the table (Figure 1). Ten cards is an arbitrary number but it creates a big enough set for various possibilities to be available and it also reinforces the very important notion of ten. The remaining cards are placed in a pile face down.

Figure 1. Set-up for Make 10

The first player looks at the available cards for the best combination of cards to make ten, that is, the combination which uses the greatest number of cards. The player removes these cards making certain to articulate the number sentence they have created. This articulation is important as it acts as a verbal check to the other players that the number sentence is correct and also it gives an insight into the strategy used to create that sentence. For example there is a point of difference (and therefore of possible assessment) if one student is using a 'count all' strategy and another is using a 'count on from the biggest number' strategy and yet another, a basic facts strategy ("I know 4 and 4 are eight and two more makes 10").

The first player lays the collected cards in front of themselves and replenishes the set of ten cards from the remaining card pile. The second player then has the opportunity to select a combination to 10. Play continues until all of the cards are used or there are no combinations that add to ten. It should be noted that the ten-card itself totals ten and that the correct articulation of this "combination" is "ten equals ten".

Although at the conclusion of the game the winner is the person who has collected the greatest number of cards, there is a little more mathematics which can be extracted from the activity. The students must also count up their sets of ten to tell how many they have in total, that is, "I have six sets of 10, so that's 60."

Once this basic game has been established variations, such as making the target number 12 or 13 (bridging the ten) and the capacity to use multiple operations ("7 add 6 take 3 = 10") can be added.

Make 10 is a game which belongs in the mathematics classroom; it is firmly rooted in mathematics as it encompasses the mathematical proficiencies (ACARA, 2013) of fluency, problem solving and reasoning, and content from the Number and Algebra strand with making and bridging ten, subitising, partitioning and addition. It also has an element of luck in the how the cards are drawn from the main deck to create the row of cards in play, but there is a strong element of skill required to exercise the mathematics to maximise the number of cards to get to the total of ten.

Combo

This game requires a set of 45 cards which range from the lowest "value" card which has the digits 1, 1, to the highest "value" card with 9, 9 (Figure 2). Alternatively, a set of double nine dominoes can be used.

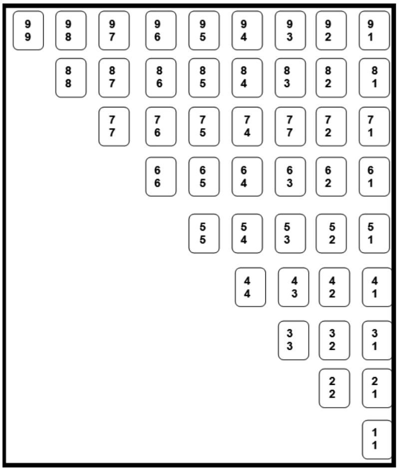


Figure 2 Combinations for the Combo pack

The pack is shuffled and each player is dealt five cards with the remaining cards placed in the middle of the table. In the early stages of developing an understanding of the game it is a good idea that the students arrange the cards face up, this allows students to help each other if required. Generally it does not take long before the students become quite strategic and wish to conceal what is in their hand from each other.

The top card from the remaining cards in the centre, is turned over and this becomes the key calculation card. For example if the 6, 3 card is turned over, the players now use those two digits and any of the four operations to create a new number. Therefore that card could be worth 9 (6 + 3), 18 (6 x 3), 3 (6 - 3) or 2 (6 ÷ 3). The first player to the left of the dealer then looks at their cards to match a card from their hand to the key calculation card and can use any of the four operations (not necessarily the operation used to provide the total for the key calculation card) to match the totals. It is essential that the player articulates how the cards match. For example the player may state "6 divided by 3 is 2 and 4 take away 2 is 2". This card now becomes the key calculation card which the next player must match.

If a card cannot be matched to the key calculation card by the next player, the opportunity moves to the player whose turn is next until all players have failed to match the card. If this occurs, the next pile on the remaining cards pile is turned. The winner is the first person to discard all of the cards in their hand.

Is this a game that engages and has the capacity to embed learning (Parsons, 2008) with opportunities for reinforcement and practice of skills, acquisition and development of concepts and development of problem solving strategies (Ernest, 1986)? Most certainly in the experience of this writer and others, this is a game which although is sometimes challenging to introduce, has the capacity to engage students and develop a fluency which make the challenge one worth taking. The number of calculations in which the students engage, the articulation of their number sentences and the strategic thinking employed are all strengths of this game. There is no doubt that there is an element of luck in the cards which can be dealt to an individual, as 23 of the cards lend themselves to all four operations (as in the case of the 6, 3 card, which allows for four different answers) whereas 22 cards are limited to three answers (for example 9, 5 has only the possible answers of 4, 14 and 45). However, the factor which will more often than not determine the winner will be the person who has the greatest flexibility and fluency with numbers.

Fraction Match

The aim of Fraction Match is to collect and place down combinations of cards which equal one. The cards have a number of different fraction types (common fractions, decimal fractions and percentages) and the representations on the cards can vary (area models, line models, set models etc.) (Figure 3). The fractions contained in the set of matching cards, and therefore the complexity of the game, can be changed to suit the needs of the students.

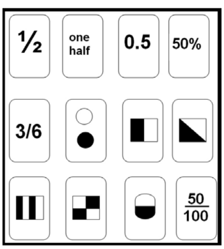
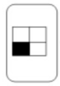


Figure 3. Some possible fraction matching cards for one-half

The student may make up combinations in any way and with any combination as long as the total is exactly equal to one. Therefore  and **0.5** would be an

acceptable combination as would be  + ½ + 0.125 + 12.5% .

Each player is dealt seven of the fraction matching cards with the remaining cards going into the centre of the desk. A card is taken from the top of the deck and is turned face-up. The person to the left of the dealer can pick up the card that has been turned over or pick up a card they have not seen from the pile. When choosing the card the player is looking to add cards to their hand which will help them build combinations to make one. The player must then discard a card from their hand so that they only ever have seven cards or less. The next player does the same. The winner is the first person to lay all of their cards down in combinations that make one.

There is decidedly an element of luck involved with the initial dealing of the cards and the order in which the cards are revealed from the pack, but the mathematics behind the ability to operate with fractions in their multiple representations, is crucial to being successful.

These are just three activities using cards from the very many which are available. Each has been shown to generate enthusiasm, add variety to a mathematics classroom, encourage mathematical discussion, promote co-operative learning and they provide visual representations of problems through manipulative operations. Just as importantly they embed learning and provide many opportunities for reinforcement and practice of skills, acquisition and development of concepts and development of problem solving strategies. They also all belong in Parsons' (2008) broad category of "luck with skill games", the favoured type of games.

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GROUPING: SUCCESSES, SURPRISES AND CATASTROPHES

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The Australian Mathematics Curriculum emphasises the need to develop mathematical creativity to foster deep learning. This paper extends our understandings of how progressive changes to group membership over time can increase student inclination to explore unfamiliar challenging problems. It reports on the successive changing of group membership in two different situations and uses this to identify key features in what occurred. Like in any classroom, there were successes, surprises, and catastrophes.

Promoting Creative Mathematical Thinking: A Need

The Melbourne Declaration of Educational Goals for Young Australians http://www.mceecdya.edu.au/mceecdya/melbourne_declaration,25979.html and the Australian Mathematics Curriculum (see <http://www.australian curriculum.edu.au/Mathematics/Rationale>) require the development of creative, innovative and resourceful problem solvers. Enabling such activity in mathematics classrooms includes providing opportunities for students to enter the Space to Think (Williams, 2005) which is a state of flow in which students lose all sense of time, self, and the world around because all their energies are focused on creatively engaging with the task at hand (Csikszentmihalyi, 1992).

My research shows that it is resilient (optimistic) students who enter that flow state. 'Optimism' is an orientation to 'failures' and 'successes' (Seligman, 1995). Optimistic children perceive failure as 'temporary' (able to be overcome), 'specific' (to the situation at hand), and 'external' (can be associated with factors beyond their control). They perceive successes as personal (achieved through their own effort), permanent (able to be achieved again), and pervasive (internalized as characteristics of self: 'I did this, I am good at this').

Creating conditions for creative thinking includes providing accessible tasks that can be undertaken using various mathematical ideas, and representations. These tasks need to contain opportunities for students or groups to discover and want to explore mathematical complexities that were not apparent to them at the start of the task. They also include autonomous and spontaneous student action so the teacher does not hint or tell but rather asks questions to elicit further thinking. Conditions for flow include students or groups spontaneously asking a question that takes them outside their present understanding and into the Space to Think. 'Same pace of thinking' groups is one of the key aspects of My Engaged to Learn Approach to problem solving (Williams, Harrington, & Goldfinch, 2012) within which students work in small groups (3-4 students) with problem solving tasks. Each group reports to the class at regular intervals with a different reporter each time. 'Priming' involves the groups brainstorming what to report, the reporter practicing this report, and the group refining it to fit what they want. This is a key element of the approach.

It is not always easy to engineer conditions for flow in lessons, and sustain it once it occurs. Crucial to optimising these condition can be changes purposefully made to the composition of groups in the class over time. This paper extends the thinking about groups beyond my previous ideas on how to compose groups (see for example: Williams, Harrington, & Goldfinch, 2012) to a focus on how to successively change group membership over a period of time to build the optimism of particular class members.

Change Sequences

In this paper, two different instances of this changing of group composition across a sequence of tasks are reported. These sequenced changes have been named: a) From Self-Focused to Task-Focused; and b) Prioritising the Majority. Each grouping sequence is discussed in terms of the dilemmas, reflecting on how to proceed, the result, the next step, and reflecting on the outcome.

From Self-Focused to Task-Focused

This three task sequence was intended to shift Tom (pseudonym) from a non-optimistic student who perceived success came from external judgments of his mathematical worth to an optimistic student who takes on mathematical successes as attributes of self: 'I can do these I am good at this'. The three tasks across which the composition of Tom's group was changed are now briefly described.

Task 1

Using all 14 tiles (each time), make as many different flat 'filled' rectangles as you can.

Repeat using 12 tiles. Have you found all possibilities? Make an argument for how you know you have them all. Do more tiles make more rectangles? Why or why not? Select a number of tiles between 16 and 45 to make as many rectangles as possible using all each time. Explain your process of thinking.

Task 2

A large fish drawn on the board has an arrow pointing to the head stating 'the head is as heavy as four tails' and an arrow pointing to the body stating 'the body is as heavy as the head and the tail together'. Find all you can about the maths of such fish.

Task 3

Use four of the digit four, and any number of the following

+ + - - x ÷ / ( ) √ ² .

to make each of the whole numbers from 1-20. Then look for ways to find them all as fast as you can. Explain.

During Task 1

Tom was in a composite Grade 4-6 class. Tom (Grade 5) was grouped with Billy (Grade 4) and Sammy (Grade 5). As shown from the excerpt of group discussion before the last of four reporting sessions, Billy (Grade 5) has begun to develop an idea but Tom argued that it was his turn to report and he was going to report his own ideas not the ideas of the group. This was contrary to class 'rules' and a disagreement ensued. The majority of the group discussion time was taken up arguing and the group did not have time to refine Billy's ideas. Group opportunities to enter the Space to Think were inhibited by Tom's focus on self:

1.	Billy	[trying to gain Tom's attention] *Tom- Tom- Tom* every time you get a number that can't be divided by three- two (pause) you are bankrupt- you can't find any more
2.	Tom	I am not going to say that
3.	Billy	Yes you are
4.	Tom	No I'm not
5.	Billy	Tom (pause) you need to say it
6.	Tom	I know you are priming me but I don't have to say it ... I am saying what I have been thinking.
7.	Billy	We need- no the group tells you what to say
8.	Tom	Yeah but that doesn't mean I have to say it

Billy had found that if you continued dividing a number and using the answer for the next step, you could not go any further once your answer would not divide by two. He thought the number of divides gave the number of rectangles. This is not yet correct but could have led to rich mathematical discussions of divisibility. Billy presented his idea to Sammy and Tom [Line 1] and began to summarise for Tom. Lines 2-4 show the types of disagreement between Tom and Billy that continued until almost the end of the reporting session. Both Tom and Billy were aware of the group rules [Lines 6, 7] but Tom disregarded them [Line 8]. He was too focused on reporting his own ideas so they could be valued by others including the teachers. He did not consider Billy's idea and whether it always worked. Tom eventually stated: 'It's my time to report and I am going to say what I want to say'. In his post lesson interview, Tom showed he did not have anything new at that stage:

"Ah I was thinking-ah well (pause) I was thinking um- oh- well (pause) um I was thinking ah there must be other ways to do it than just following a pattern- and I know you should follow patterns but there has to be a different way ..."

Tom wanted to report something original. He was focused on himself and not the task. It did not matter that Billy had presented the start of some idea that could turn out to be very useful, to Tom that idea was around patterns and patterns were something that many groups focused on. He wanted to be *different* rather than solve the task.

Preparing For and Undertaking Task 2

To increase Tom's likelihood of gaining task focus, Tom was grouped with two Grade 6 students Natasha and Ken who had previously collaborated to build insights. They were both focused and calm and it was considered they would be able to stop Tom from exhibiting 'take over' activity and make him conform to group reporting rules. The group did calmly discuss ideas, and they quietly used body language to deter Tom's attempts, initially, to 'take over'. Tom listened and contributed by asking when he did not understand. The group primed Tom to present a clear report that contributed to class knowledge. Tom expressed his interest in this task after Task 3: "You can just draw something so small with so little detail and find out so much about it and end up [small laugh] (pause) just like you have known it forever." This task contained an element of surprise that engaged Tom with the mathematics. His awareness had been raised that you could learn from these tasks and they may not be as simple as they first appeared. Tom was pleased with the result but as researcher and teacher, I wanted more. Although Natasha and Ken had entered the Space to Think and created new ideas as a result, Tom had not, he had listened and learnt from the other two. He had not contributed to the newly created ideas. My next question became, 'who could I group Tom with who was a creative thinker but thought at a similar pace to Tom, and would be able to control any 'self-promoting' activity? If I could place him in such a group, would he now focus on the task and contribute new ideas?

Preparing for and Undertaking Task 3

Tom was grouped with Gabriele, a student who I considered would be able to 'control' any 'take over' activity and could think creatively but not at a fast pace. This was expected to give Tom more time to think. It was hoped that Tom and Gabrielle would create new ideas together. The other two students in the group had not previously contributed creative ideas but had not disrupted their groups. In this group, they would hopefully have opportunity to learn and also experience the creative development of new ideas by others (as Tom had in Task 2). Tom was the final reporter for his group. In the previous reports, Alf from another group explained: 'Well four over four is one whole so that's just like saying one (pause) and four times four plus one you would get seventeen'. Tom reported his excitement at hearing this in his interview: 'When he said four over four and it is the same as one just that sentence just flung me [intense, twirls hand] like quickly in my mind *ahhh* I could use that". Tom's idea was to use two of the four fours to make a multiple of four then the other two fours as: "Four over four- ... that's one okay ... so that means that using a whole you can go ... minus 1 or plus one ... we could get these numbers ... eleven thirteen fifteen seventeen and nineteen and twenty". Gabriele had either worked out that Tom would not be able to make each multiple of four, or wanted to know how he was doing this but Tom just kept repeating his idea about plus or minus four over four. Gabrielle insistently requested elaboration as she tapped the table in front of Tom to gain his attention and asked with intensity: "*how how how how* are you going to get seventeen and nineteen twenty?" When Tom again focused on the end not the stem of the calculation, Gabriele did not give up, she picked up and handed the pencil and paper to Tom stating: "So if you were going- if somebody asked you to make- to get answers for every single number [ruling up sheet] ...". Gabrielle's request for elaboration led to Tom's revised ideas which limited the stem to 8, (4+4) and 16, (4x4). He elaborated on the reason for his changed position: "something like 12 (pause) you won't be able to do it because four plus four plus four is one of the only ways to get to twelve and ... [there] will be five fours". Tom's ideas were refined through Gabrielle's insistence for justification and elaboration. He had located some correct possibilities but did not see new possibilities for making the stem using two fours and other operations.

The Outcomes

In his final interview after Task 3, Tom showed he now considered himself as a problem solver after 'seeing' his activity over time. He described how his Mum had told him he was good at maths but he had not believed her: "but when I have gone through with these [tasks] (pause) I have believed that I am a bit smart". Tom has begun to shift from perceiving success as external—others acclaiming his ideas—to making his own assessment about his maths ability by reflecting on his task activity. Tom had begun to perceive Success as Personal. Tom changed from self-focused (Task 1), to group focused (Task 2), and task focused (Task 3). He experienced surprise as complexities became apparent in what had appeared to be simple (Task 2), and displayed positive affect during his creative thinking leading to insight (Task 3). Gabrielle's insistent 'push' for elaboration was crucial to Tom extending his understanding of the detail of the ideas he was putting forward. Without this, Tom would not have refined his initial ideas.

In Tom's case, creative mathematical activity accompanied by high positive affect was associated with increased indicators of optimism as expected (Seligman, 1995). Indicators were tentative not strong. This grouping sequence highlights the important role group composition played in changing Tom's orientation to problem solving. This student who had required external affirmation to perceive himself as successful inhibited opportunities for his group to think creatively in Task 1. The non-optimistic indicator he possessed (perceiving Success as judged Externally) created an unproductive interaction because Tom was not focused on the task but on 'showcasing' himself. By successively grouping Tom to inhibit this non-productive activity, then giving him an opportunity for undertaking creative thinking himself, there is now another optimistic student in the class to promote rather than inhibit creative thinking.

Prioritising the Majority

One of the most surprising and devastating findings in my study of the role of optimism in collaborative problem solving has been the inhibiting of creative activity by high performing confident students *who are not willing to 'step outside' what they know mathematically to explore new mathematical ideas* (Williams, 2012). These are students who perform near the top of the class on class tests and are perceived by the class and the teacher to be 'good at maths'. They differ to other high performing students in the way they learn, and in the way they perceive their mathematical ability. Where optimistic high performing students describe how they learn using active words like 'thinking things out' these high performing non-optimistic students describe learning as 'listening to the teacher, reading books, and using the internet'. They judge their mathematical ability using external judgments made by others whereas optimistic high performing students primarily judge their mathematical ability through self assessments of their ability to work things out. Now here is the dilemma! In one class there were four of these high performing confident students who were unwilling to explore new ideas. In the first task, they were in four of the seven different groups. In each of these groups, other students did not have opportunity to enter the Space to Think because these non-optimistic students took up all the talk time with mathematics that was already known and cut off the talk of any student who raised a question about a mathematical complexity that was new to all group members.

What to do? These students were stopping opportunities for others to think! What would happen if we grouped them all together? This would give other groups an opportunity to explore new ideas that arose but what would happen to this group of high-performing non-optimistic students? These students are not optimistic because they perceive success as external (they learn only through external sources not the reorganisation of ideas, and they draw on external sources as evidence of their mathematical ability). We grouped these students together and guess what happened!

Many other groups did begin to think creatively, and the group of non-optimistic high performing students did not 'step outside' their present understandings. They were unable to progress with the problem solving task. What to do now though? It is the obligation of the teacher to develop the potential of all students. This had been addressed in other groups, but what about this group? We decided to leave them together and see what we could do by asking them more questions and keeping a watch so we could accent anything useful that came into their conversation. There had been no progress with this group one task later and the following year three of them were in the same Grade 5-6 class. We decided to continue to group them together and see if we could find a person who had demonstrated they thought creatively who was strong enough to continue to do so '*out loud*' despite the inhibiting activity of these three students. We were surprised and delighted at what occurred in the next task.

One of the three non-optimistic high performing boys was absent. The other boy we decided to add to this group was intensely interested in mathematics and willing to explore new ideas and sufficiently 'strong' to make sure that his ideas were heard by *and used* by groups he had participated in. Adding this student to make a group of three worked extremely well. The group listened to the new ideas presented because this boy was persistent in presenting them, he was willing to explain what he was thinking to the others, and he was willing to answer their questions as they developed an understanding. Through his passion, and determination, it appears that this creative thinker was able to present himself as a mathematical authority: 'expert other' (Vygotsky, 1978). This fitted how these high performing, confident, non-optimistic students learnt: *taking in* information from an authoritative source. In their reports to the class, each group member was excited about what the group had found, and able to communicate it clearly to the class.

What we learnt was: it was possible for two non-optimistic high performing students grouped together with a forceful, creative student (willing to take time explaining his ideas) to come to the types of realisations Tom had developed in Task 2—that there was more to these tasks than initially apparent, and that presenting ideas that interested the class was accompanied by positive experiences. Further research is needed to find whether these students do become optimistic over time when placed in such groupings. In addition, we need to know: 'was the *sequence* of groupings necessary or would this final grouping have worked straight away? In other words, did these high performing non-optimistic students need to be *surprised* that other groups (of students they perceived to be not as 'good at maths' as they were) developed more creative ideas (than they did) before they would be responsive to new ideas from another student? Was this sequencing needed for this responsiveness to occur?

What Have We Learnt?

What can we take out of these experiences as teachers? New ways to begin to think about building the creativity of members of your classes. Not only are the criteria for the composition of groups important but there is also further thinking to do. By attending over time to the personal characteristics of group members, and selecting those who might best be able to productively channel the activity of others, there is opportunity to raise the awareness of students about the complexities that lie hidden within what appear at first to be simple tasks. There is also opportunity for students to vicariously share in the creative successes of other group members by becoming sufficiently familiar with the new ideas to be able to communicate them clearly to the class, and experiencing the pleasure associated with class exclamations about what has been found. This is optimism building.

Acknowledgements

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Mathematics
of Planet Earth 2013

- Peer Review

ALGEBRA AS STORY TELLING

Mike Clapper

Australian Mathematics Trust

This presentation describes a framework for the introduction and development of algebraic thinking which develops in students the understanding that algebra is about 'things that happen to numbers' in a narrative context. Whilst it draws on some well-understood pre-algebraic pedagogies such as machine games and back-tracking, it develops these into a fuller picture of algebraic processes using the technique of 'unambiguous labelling' which relates every algebraic expression (or equation) to the story which it tells about numbers. Many examples will be given of practical activities which will allow students to use their emerging algebraic skills to explore patterns and develop algebraic thinking.

Preamble

How often do we hear a student (or their parent) say that they were going along perfectly well in Mathematics until they hit High School? What is the leap required in High School which makes so many students lose confidence in their capacity to be successful in Maths? The answer, as every High School teacher knows, is Algebra.

This paper will argue that this does not have to be the case. Algebra is important but it does not have to be mysterious. There are powerful ways to assist students to understand algebraic concepts, but, all too often, we resort to teaching it as a set of arcane skills that 'just have to be learnt'. Not surprisingly, for all but the most able students, these skills do not 'stick' very easily and, when it comes to applying them to worded problems, there is a collective moan which reflects the gap between being able to replicate a given rule and truly understanding it.

Ironically, before they reach High School, most students have dealt with a range of algebraic (or pre-algebraic) ideas in Primary School, often without being aware they were doing something which was supposed to be difficult. In fact, we can all probably think of students who, in Primary School, could deal with $\square + 7 = 11$ and have no difficulty figuring out the number in the box. However, the same student may well be bamboozled by $x+7=\mathbf{11}$ in Year 11 because as soon as they see the pronumeral, they think, that's algebra, and I can't do algebra. They are forced into a mode of thinking which is about 'What are the rules I learnt for this situation?', rather than 'What does this expression mean?'

Let us think, just for a moment, where High School algebra typically begins. Most textbooks start with the collection of like terms and algebraic simplification. Students are expected to learn how to gather terms and before very long are dealing with expressions like $2a+3ab-3a-ab$ and expected to understand that these terms can be rearranged according to a set of rules. There is almost no sensible application of this knowledge at this stage and this greatly impacts on the teacher's capacity to make it relevant and meaningful to the students, without resorting to the highly dubious 'apples and bananas' approach. The obvious conclusion is that this is the wrong starting point.

Every algebraic expression is a story about numbers. So we need to start with telling this story in a way which makes sense to students. Actually, this process has already started in Primary School. Ideas such as skip-counting (introduced in Grade 1) and simple machines are number stories with which students are familiar from about Grade 3. Interestingly, the context of the machine is a help, not a hindrance, to the thinking. There are very few students in Middle Primary who cannot work out what is happening in a simple one-stage machine (that is, one that uses just one of the four operations). By Upper Primary, most students can deal with two stages. So they are doing algebra. If we don't obsess about how they set it out, we would expect that a typical Year 7 student would have little difficulty with a problem like:

Jane saves her pocket money for three weeks and her mum gives her an extra five dollars which means she has just enough to buy a Lego set which costs $29. How much pocket money does Jane get each week?

For most students, the context here is helpful. They can visualize what has happened and the order in which it has happened far more easily than when trying to solve the context-free equation $3x+5=29$. In fact, the teacher who insists that they get no marks for solving this question unless they can write down the equation representing it is missing an opportunity.

As teachers, we know why it is important to be able to write this down algebraically, what we need is a teaching sequence which starts with the intuition of the students and gives them a way of representing their thinking over which they have control, and through which they can see that they are now able to solve more difficult problems which might be beyond their intuition.

A Teaching Sequence for Algebra

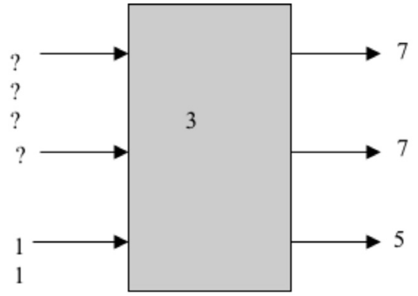
Much of the material presented here will be familiar to many teachers. What may be less familiar is the concept of 'unambiguous labeling' which is at the heart of developing algebraic understanding. The emphasis is on language, both the language in which stories are told (including the metaphorical component) and the formal language of algebra. The teaching sequence is:

- Machine Games
- Labeling machines
- Undoing machines (intuitively)
- Building larger machines by building smaller machines (number stories)
- Unambiguous labeling
- Algebra stories to Box stories
- Detective stories
- Solving the 'Crime' – backtracking
- Equations
- Patterns and applications

We will now explore each of these in a little more detail.

Machine Games

These will be familiar to most teachers and are now a part of the Australian Curriculum (HREF1).

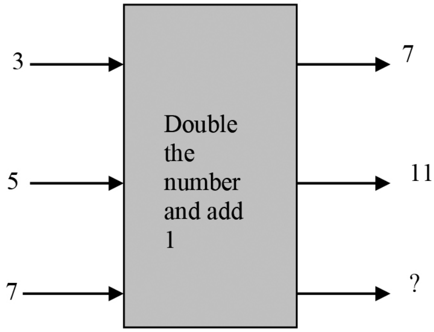


The important part of using mathematical machines is that they encourage students to think of algebraic processes (though they will not use that term) as 'things that happen to numbers'. Hence, from the outset, the emphasis is on the process rather than on the particular numbers which happen to be either input or output.

Students in middle primary will mostly be able to describe a single operation machine, once they get the idea of the game. Machine games can be played in a very interactive way with students inventing the machines and other students providing inputs and guessing outputs.

Labeling Machines

Of course, the critical element is the student's capacity to describe what is happening in words. This must be done in a general way without reference to particular input numbers.



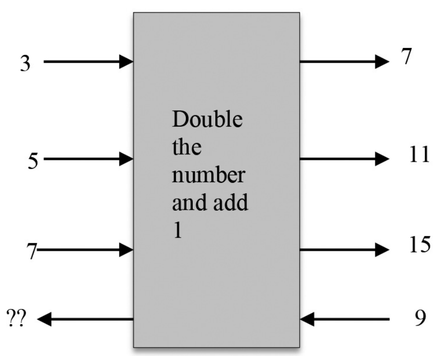
It is good if they make reference to 'the number' and this should be encouraged rather than just 'times 2, add 1'.

There may be more than one correct description, such as 'double the number and add 2' being the same as 'add 1 and double the number'. This will be a useful and interesting item for discussion, but is not critical at this point. Two stage machines can usually be managed by Grade 5 and 6 students and some students may be ready for different operations such as squaring, though these should not be a part of the standard set unless they are also capable of handling square roots.

Students may want to use the operation symbols rather than words and maybe some will want to abbreviate the word number!

Undoing Machines

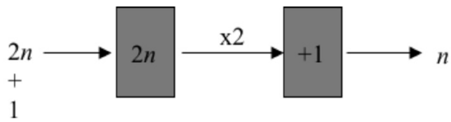
It may be worth asking whether students can 'undo' a given machine by predicting the input when they know the output. This will provide teachers with an awareness of the intuition which students might possess about inverse operations.



However, this may prove quite difficult for machines with more than one stage. If you believe that your students can deal with two-stage operations, you can use a 'dressing – undressing' metaphor to explain why the order of operations needs to be reversed. Undoing machines is dealt with more formally later in the unit.

Building Larger Machines by Building Smaller Machines (Number Stories)

It is good to have some class discussion on how much more difficult it is to figure out a two-stage machine than a one-stage machine. If we were to add more stages, it would become even more so. Hence, to analyse something more complicated, we need to break it down into smaller steps. So, we are interested in what is happening inside the machine. This leads naturally to the idea of building a machine from smaller components.



I prefer to put the operators in the boxes, not the numbers. This is because we are trying to emphasis that the numbers are incidental, it is what is happening to them that is important. We now need to justify why we want to come up with an efficient labeling scheme for our machines.

Unambiguous Labeling

We now get to the heart of the matter. I tell the students that they are all workers in my mathematical factory, building machines. If we have a large order we want to make sure that each worker labels the machines in an identical fashion, so we need to agree some rules for labeling the machines. The rules are:

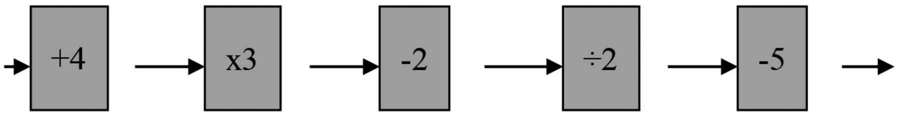
1. **Addition and subtraction on the right** *(That is, when they get to a box which is an addition or subtraction, they just use the + or – sign on the right-hand side along with the number. eg* $n+6$*)*
2. **Multiplying numbers on the left** *(brackets around everything except a single unknown) (In this simplified set of rules, because multiplication is the only thing which goes on the left, the sign is redundant – this causes surprisingly little difficulty! eg* $3n$ *or* $3(n\text{-}1)$ *)*
3. **Division underneath** *(When we get to a division, we draw a line under all we have written to that point, then put the dividing number underneath)*

   Later on - squares and square roots *(These are not difficult when students are ready for them. Again with the use of brackets around what we have done so far, the square sign goes in its normal position, which is not the same as any of the above. The square root sign acts as its own bracket, like the division line eg* n^2 *or* $\sqrt{3(n-1)}$ *)*

This is not a complete algebra system, but it is sufficient to write a unique label for any set of boxes. We do not worry about the simplification of algebraic expressions at this stage.

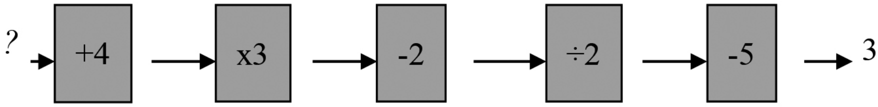
Algebra Stories to Box stories

Armed with these rules, the students should now be able to turn a box story into an algebra story (or label) and, after a bit of familiarization, to do the opposite. This needs time and a gradual introduction of levels of difficulty, but before long they should be able to turn this: $\frac{3(n+4)-2}{2} - 5$ into this:



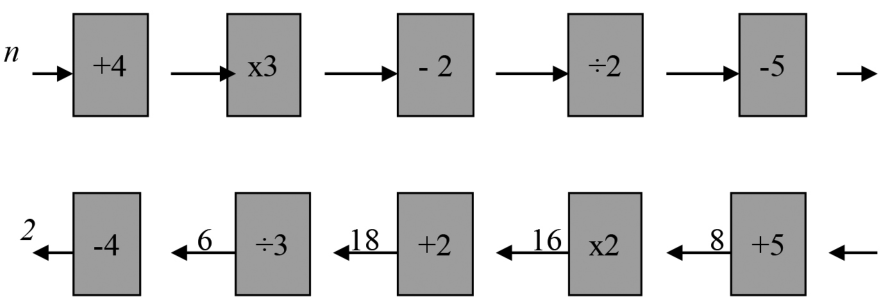
Detective Stories

However, their interest will soon wane if they are doing nothing with these stories, so we need to introduce stories with endings, or Detective stories, where we know what the final result is (the crime) and want to find out what happened along the way.



Solving the 'Crime' – Backtracking

We introduce a systematic way of backtracking through a box story



I like to get students to write the reverse box story underneath, but the more confident ones always think they can do without this (and they usually can!)

Equations

Now, of course, we want to complete the process by starting with the equation, turning it into a box story and backtracking to find a solution. I have found very few students who cannot successfully manage equations with several stages using this approach. The key is that the unambiguous labeling makes the stories easy to unravel. Better students can often do 8, 10 or 12 stage equations in their heads after some exposure to this method.

Patterns and Applications

Using this technique, we can now put our algebra to practical use by investigating some pattern questions (such as matchstick patterns – e.g., the number of matches needed to create a certain number of boxes), using machine thinking to identify the connection between variables (number of matches, number of boxes) and writing the connecting equation using the labeling technique. From here we can solve questions in which either of the variables is unknown. This means that students, in their first experience of algebra, can see its power to provide generalized results which they can put to practical use.

Limitations and Power of the Method

This not, of course, a complete algebra. The most obvious limitation is that it is confined to solving equations where the pronumeral is in one place. However, this is a powerful starting point for more difficult equations, because the method of solution (in all cases) is to rearrange the equation to get the variable into one place, after which it can be backtracked. Hence, when we come, for instance, to completing the square, the rationale for what we are doing is precisely this. This applies to trigonometric equations and even to matrix equations. This seems to me to justify this as being the right place to start a high-school algebra course.

Another important element in this approach is that, by using a narrative (i.e., story-telling) approach, we may be able to appeal to students who find the conventional, purely symbolic approach, off-putting. This may include, but will certainly not be limited to, indigenous students.

Embedding This Thinking Into More Advanced Topics

My experience has been that students who start their algebraic journey with this approach willingly refer back to it in later years. I have often introduced algebra as story-telling to Primary students who I have taught again in Year 11 or 12. The shared narrative is always helpful when dealing with more advanced topics. It is much easier for students who have a narrative understanding to deal with an algebraic expression such as a formula, because they know how to interpret the 'story' implied in the formula. Indeed, they can use backtracking to rearrange the formula if required. Ironically, although backtracking is confined to one unknown, the familiarity with 'story' makes it far easier to discuss more complex algebra expressions in meaningful ways.

In the session, I also provided worksheets, a number of practical applications and the original powerpoint. If anybody is interested in copies of either, they should contact me on mikec@amt.edu.au or go to my googledrive:

https://drive.google.com/folderview?id=0B50ETVdn3birQ2VoeGo2cEFTb1E&usp=sharing

I would also be interested to receive feedback from anybody who has tried using this approach in the classroom.

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THE BIG IDEAS IN NUMBER: 1 SMALL STEP FOR A SCHOOL, 1 GIANT LEAP IN MATHEMATICAL REFORM

Margaret Dolan, Amanda McLean and Robyn Trzeciak

River Gum Primary School, VIC

Picture this... A primary school in the suburbs of South East Melbourne. A school of children ready to learn and teachers ready to change their mathematics practice. Staff recognising the need for change, and actively asking, what could be done to improve student outcomes in Numeracy? NAPLAN data is low and a Scaffolding Numeracy in the Middle Years (SNMY) (Siemon, 2006) Assessment completed by Grade 3s to 6s, places most students in the Learning Assessment Framework (LAF) Zone 1. Our choices appear simple; something needs to be done and quickly! This is where our story begins...

Introduction

Here at River Gum Primary School we have designed and implemented a whole school framework for teaching mathematics based on the Big Ideas in Number research of Professor Dianne Siemon.

School Context

River Gum Primary is a school with approximately 480 students in Hampton Park, in Melbourne's South East. River Gum has a transformational pedagogical vision which drives change initiatives. A variety of programs are on offer to students at River Gum. Students are encouraged to manage their own learning and be responsible for their own behaviour. Our core business is to focus on teaching English and mathematics.

Our 'One Small Step': Identifying the Need for Reform

At River Gum in 2011, we identified mathematics as a school priority, as reflected in our Annual Implementation Plan for 2011, 2012 and 2013. Our NAPLAN (The National Assessment Program - Literacy and Numeracy) data has shown it was an area that we needed to focus on. Both NAPLAN and school wide assessments using the Scaffolding Numeracy in the Middle Years (SNMY) (Siemon, 2006) as an indicator placed us well below average results. (This data analysis took place in consultation with Professor Diane Siemon).

During an open forum on mathematics, our classroom staff identified a lack of confidence in their ability to deliver a quality mathematics program. Staff expressed concern about students who were progressing through our school with gaps in their mathematical knowledge which was hampering their progress. Staff were interested in creating a school wide structure for teaching mathematics, which included a shared understanding of what good practice is.

In 2012, through the Casey Numeracy Network, discussions took place about using common assessment across schools involved in the network. We were very interested in the work of Professor Di Siemon of RMIT University, in particular, her Scaffolding Numeracy in the Middle Years assessment tool (Siemon, 2006). Professor Siemon was working in partnership with the network. We had the privilege at River Gum of seeing some new work she had been developing. This work centered on the development of Developmental Maps which contained the Big Ideas in Number concepts. We expressed our desire to create a framework that allowed teachers to have a deep understanding of what mathematical concepts were important at each stage of development. Thus, a University-School partnership was born. Professor Diane Siemon has worked in partnership with River Gum Primary School, in terms of allowing us to use her research to create a framework for teaching mathematics, and offering her time and expertise in guiding us in the shaping and direction of this framework.

"Houston…We Have a Problem!"

In 2010, River Gum Primary School had been appointed a new principal. She quickly assessed the school environment as being disorderly and chaotic. The principal drew on her knowledge of Zbar, Kimber and Marshall's (2010) '10 Preconditions for School Improvement' (refer to Figure 1), to establish a safe and orderly learning environment so that school improvement could occur. As stated by Zbar et al. (2010, p. 5), "the absence of an orderly learning environment is usually the first thing noticed in an underperforming school, and is the major impediment to improvement and change. In addition, the establishment of such an environment, and the consistency of staff behaviour on which it depends, is commonly the key initial strategy for the leadership team in turning the school around".

The Executive Leadership Team headed by the principal then began to focus on developing an agreed school wide approach to student behaviour, building leadership capacity and teacher capacity, with a focus on student learning outcomes.

Figure 1. Preconditions for school improvement. Adapted from V. Zbar, R. Kimber, & G. Marshall, 2010, *Getting the preconditions for school improvement in place: How to make it happen. Seminar Series 193.* East Melbourne, Victoria, Australia: The Centre for Strategic Education (CSE): Mercer House.

1.	Strong Leadership that is shared
2.	High levels of expectation and teacher efficacy
3.	Ensuring an orderly learning environment
4.	A focus on what matters
5.	Building teacher and leadership expertise
6.	Structure teaching to ensure all students succeed
7.	Using data to drive improvement
8.	A culture of sharing and responsibility
9.	Tailoring initiatives to the overall direction of the school
10.	Engendering pride in the school

Department data (NAPLAN) was used to establish a base line of student academic achievement. This base line data was then compared to current NAPLAN data to establish the area/s of most concern. Professor John Hattie's research on Effect Size (ES) was used to equate a progress score to measure any improvement and growth. Hattie describes Effect Size as providing "… a common expression of the magnitude of study outcomes for many types of outcome variables, such as school achievement" (Hattie, 2009, p. 7). An ES gave us a scaled score to compare and discuss the student NAPLAN data sets. According to Hattie (2012), an ES of 0.4 is considered to be average growth (in terms of student learning outcomes) over a one year period. Anything above 0.4 is considered high and anything below is considered low. A reverse effect will occur if a negative ES is recorded.

Mathematics ES at River Gum Primary in 2008-2010 was 0.09 compared to the 2010-

2012 ES at 0.15. Both of these results are low in terms of impact on student growth. The leadership team at River Gum drew on these results as the imperative reason for school wide reform in mathematics.

Once the majority of staff were 'on board' with the changes that were going to be taking place, the leadership team created a sense of urgency for this reform to occur. This was a vital and fundamental step for all stakeholders involved. Kotter states that "When the urgency rate is not pumped up enough, the transformation process cannot succeed and the long-term future of the organisation is put in jeopardy" (Kotter, 2000, p. 60).

Our 'One Giant Leap': Creating the Tools for a Whole School Framework

In 2012 two teachers had been accepted to complete a Teacher Professional Leave (TPL) project in mathematics. Throughout the TPL project a number of resources and tools were created to help teachers build their capacity and confidence, in conjunction with intensive professional development delivered to staff around the Big Ideas in Number.

Why a Focus on Professional Development?

Professional learning for teachers is generally limited to an instructional leader presenting workshops to the whole staff, with little or no regard to the stages of teacher professional and personal development described by Glickman, Gordon and Ross-Gordon (2004). There is an assumption that all teachers who undertake the professional development are at the same stage in their own learning, and that the content presented at the last workshop was understood by all. Barth (2001) describes this approach to learning as the 'Transmission of Knowledge' model. The instructional leader presenting the workshop assumes an "autocratic style, whereby the leader tells the group members what is to be done, when, and by whom" (Barth, as cited in Glickman et al., 2004, p. 327). Unfortunately, this is resulting in teachers becoming disconnected from the ideology of personalised learning for themselves and their students. At River Gum Primary we have moved away from this model and engaged teachers in action research projects.

Key ideas of the professional development we undertook at River Gum Primary in 2012 explored the theory behind the mathematics and how it could be put into practice. The conclusion of the professional development that year was an action research project, where staff worked in groups, focusing in on one area of the Big Ideas in Number and presented their findings. The presentations addressed criteria and working questions that identified key concepts, vocabulary, possible assessments and activities, and what a lesson might look like in their particular area of focus.

This approach has continued in 2013, where staff now undertake authentic professional development. It is authentic in the sense that it is undertaken in response to student data, which informs us that mathematics needs to be a focus of teacher development because mathematics is an area of student need. Here at River Gum we aim to embed professional learning in the daily work and in the culture of our school (Department of Education and Training, 2005). Hattie discusses the premise that professional development for teachers should revolve around the needs of the students that they teach (Hattie, 2012).

Implementing the Big Ideas in Number

The research of Professor Diane Siemon looks at the skills and concepts, or 'The Big Ideas', that are essential to students' progress in mathematics. The 'Big Ideas' are a collection of four main areas that incorporate the following: trusting the count, place value, multiplicative thinking and partitioning. These are mapped in a series of four Developmental Maps. The maps cover Number & Numeration, Calculation, Pattern & Structure & Quantitative Reasoning (Siemon, 2011). The maps show how concepts link to each other and provide teachers with a valuable tool that can assist in determining where gaps may be in a student's learning, and, where to move them next.

The Crowded Curriculum

In a chapter about 'The Big Ideas in Number', Professor Siemon and her co-authors make the point that

The crowded curriculum and the lack of succinct, unambiguous guidelines about the key ideas and strategies needed to make progress in school mathematics have long been a concern of teachers. This is particularly the case for Number which is the area most responsible for the significant range in mathematics achievement in the middle years of schooling. (Siemon, Bleckly & Neal, 2012. p. 19)

The above passage describes the way that our classroom teachers reported feeling about teaching mathematics. They felt that there was such a breadth of material to cover that it was challenging to know what to focus heavily on. They also reported feeling ill-equipped to support students who were experiencing difficulty with number concepts, as identifying the skills that need extra focus was problematic. When we were first introduced the Developmental Maps that are a visual representation of Siemon's work, it became clear that the maps supplied a way of assisting teachers to clarify the order that number concepts should be introduced, and how they are inter-linked.

Using Siemon's research and theories as the basis of reform, we developed our whole school framework. Of the most significant tools developed and one of the key stages of our implementation process was the development of the Student Concept Maps (see Figures 2 and 3).

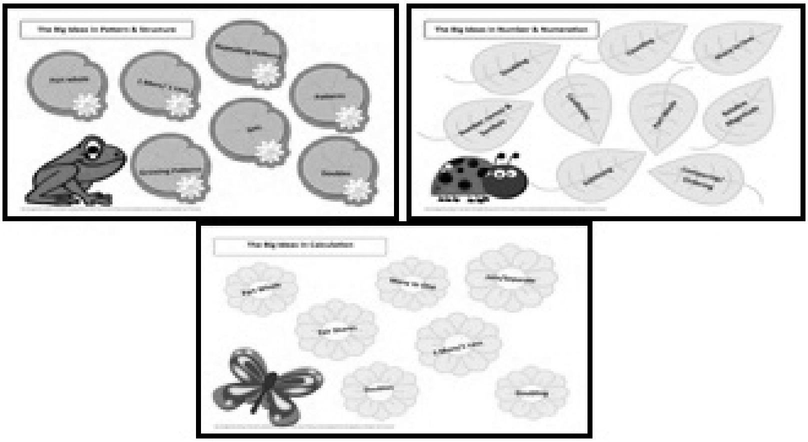

Figure 2. Student Concept Maps. Adapted by Robyn Trzeciak and Amanda McLean from D. Siemon, 2011, *Developmental maps for P-10*. Materials Commissioned by the Victorian Department of Education and Early Childhood Development, Melbourne.

The Student Concept Maps are displayed and used in classrooms. They contain all the skills covered by the Developmental Maps in a student friendly manner. The maps are easy for students to read, and allow them to track their own learning. The Student Concept Maps allow teachers to explore with students how mathematical concepts are linked and not isolated, and have provided the opportunity for the development of a common language around mathematics.

In the classroom, teachers have resource banks developed for each Big Idea in Number that contains appropriate activities relevant to various concepts. These activities allow students to work independently on areas of need or choice, while the teacher is then able to conference with individual students or work with strategy groups. During conferencing sessions, teachers are able to set goals and identify where students are at and where they are heading. They track conferences through anecdotal notes in a form of their choice either electronically or in hard copy. Student goals are identified from SNMY data and Common Misunderstandings Tools (Siemon, 2006).

Figure 3. Student Concept Maps and Resources Being Used in the Classroom.

Personalising Learning

This process allows the learning to become personalised and meaningful. Students are encouraged to monitor their own learning through the use of success criteria where they are able to 'check in' with the teacher when they feel they can demonstrate a new concept confidently. These criteria are regularly revisited. This approach is proving to be successful with teachers feeling clearer about the path their students are taking. Students feel success in their accomplishments through the success criteria and the personalised learning approach. Data are also showing that implementing Siemon's work as a whole school approach to mathematics has been a worthwhile endeavour.

Links to Research

With the implementation of new approaches and pedagogical shifts, the leadership team and staff at River Gum Primary School have felt that measuring the impact on student learning is imperative. Hattie's work on Effect Size has become the main focus to measure this impact. Hattie states that "An effect size provides a common expression of the magnitude of study outcomes for many types of outcome variables, such as school achievement" (Hattie, 2009. p.7).

What the Data Show

The graphs below show significant improvements in student data in Grades 3-6 which completed the Scaffolding Numeracy in the Middle Years in 2012 at the beginning and the end of the same year. In term 1, 2012 only 24% of grade 5/6 students achieved a LAF score of 3 or higher; in Term 4 of the same year this had increased to 56%. Similarly in Term 1, 2012 only 1% of grade 3/4 students achieved a LAF score of 3 or higher; in Term 4 this had increased to 37%.

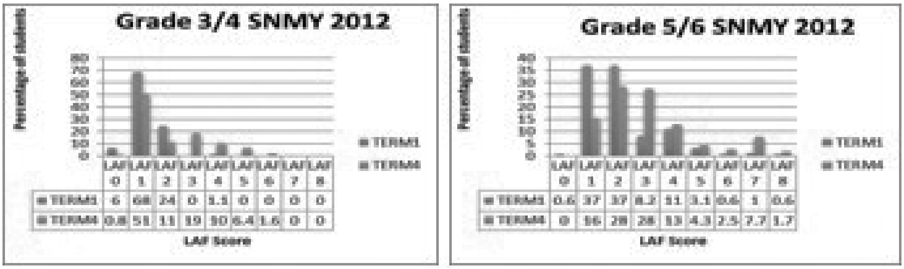

Figure 4. SNMY Results, 2012. River Gum Primary School.

Where to Next?

River Gum Primary School has undergone mathematical reform for well over a year now. Here at River Gum we are eagerly awaiting current NAPLAN data to assess and equate a current ES measure of impact.

The document *Towards Victoria as a Learning Community* (Department of Education and Early Childhood Development, 2012) outlines the need for schools in Victoria to develop whole school frameworks. This document acknowledges that many schools have ‘pockets’ of good practice. Whole school frameworks encourage teachers to work collaboratively to build their capacity and, here at River Gum Primary, we have focused our energies into creating a learning environment which fosters high expectations for both our students and our teachers.

Our new work for the end of 2013 and into the new school year of 2014 will move towards the further creation of a whole school framework for the delivery of the mathematics curriculum. Our teachers have asked for the development of an instructional model which explicitly describes what a mathematics lesson will consist of, in terms of explicit instruction, independent work, strategy groups and individual conferencing. Once again, we will be taking a further step forward in developing a shared understanding of best practice in mathematics at River Gum Primary School. Just like the whole staff forum in 2011 which set us off on this path of reform, this new work is very organic in nature, in that it has come from a need which was identified by our classroom teachers, through the analysis of assessment data.

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SPATIAL CONCEPTS AND REASONING ARE VITAL IN FULLY UNDERSTANDING PLANET EARTH

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Having students make sense of the space around them, identifying features of 2D shapes and 3D objects, developing ability to visualise images, then describing and representing them has been seen increasingly as vital for success in a wide range of careers. This area of mathematics has a strong emphasis in the Australian Curriculum: Mathematics strand 'Measurement and Geometry' and in the General Capabilities 'Using Spatial Reasoning'. This article outlines some of the research that has informed current thinking and where content appears in the Australian Curriculum.

Geometry in the Primary Years

Lehrer, Jenkins, and Osana (1998) argued that geometric and spatial thinking are important parts of mathematics with reasoning about space providing a window into issues of the mind. The Early Numeracy Research Project (Clarke et al., 2002) emphasised:

As well as being important in themselves, geometric and spatial thinking form the foundation of much learning of mathematics and other subjects. For example, teachers of older students use geometric models for arithmetic when they use grids to illustrate multiplication or circles or regions to illustrate fractions. (p. 155)

There needs to be an emphasis in classroom programs on identifying shapes according to properties, rather than just their appearance. Teachers should present plenty of examples and non-examples, and take care not to show only "prototypical shapes" such as triangles that have horizontal bases and two or more sides the same (like a gabled roof). It is important to assist the development of more sophisticated understanding where students are able to identify not only the properties of a shape but are also able to discriminate between what shapes belong in a particular class, and to begin refining their definitions. Greater exposure to manipulative and virtual materials and the physical environment can assist the development of spatial thinking. The teacher's role is to question, model correct language, and focus the engagement and thinking of students.

Properties of Shape

Among the best-known research in geometric thinking was that of van Hiele (1986). Research found that, with appropriate instruction, students develop

- visual recognition of shapes by their appearance as a whole (level 1—"it's a triangle because it looks like one")
- analysis and description of shapes in terms of their properties (level 2—"it's a triangle because it has three points").
- abstract/relational (students form abstract definitions, distinguishing between necessary and sufficient conditions—"a square is a kind of rectangle, but a rectangle is not a kind of square")
- formal deduction (where students establish theorems within an axiomatic system)
- rigor/metamathematical (where they reason formally about mathematical systems).

In junior primary years, most emphasis is on the first two van Hiele levels (Clarke et al., 2002). Success depends on students identifying the important features of two-dimensional shapes and use these features to compare and contrast shapes, for example, that students appreciate that triangles have three straight sides and that the three angles can be of various sizes. Important features of rectangles (including squares) are that they have four straight sides and four right angles and are special 'quadrilaterals'. Other quadrilaterals have some features in common with rectangles, but also differences. Students need to know the names of shapes and objects, but this is not the only focus of geometry.

Working memory is about five items which is why if you look up a phone number and someone speaks to you on the way to the phone you usually have to look it up again. Usefulness of names is that once properties of the shape are learned and connected to the name, then memory of the name carries its properties.

Revisiting the concept at different times where images which are not proto-typical are used enables the learner's concept to be appropriately expanded. It is important for students to articulate the similarities and differences between shapes and objects. Activities that can assist them in seeing the features include:

- *Shape sorting.* Put various triangles, quadrilaterals, hexagons and non-examples together. Which belong together? Name and describe features of each shape as they are sorted. Label the groups.

- Feely bag. Use a feely bag with 2D shapes or 3D objects inside. Students choose a shape by feeling and describing the shape's features, then check.
- *Shape walk.* Find a shape that has four sides, a shape that has three corners, etc. Record any shapes that you see and share this later. What is the most common shape in the environment?

Australian Curriculum: Mathematics

The required content in shape within the Australian Curriculum is found in the Measurement and Geometry Strand. At Year 1 and Year 2 the focus is very much on classifying and features of shapes, and developing their language.

Measurement and Geometry

Year 1 Shape:

- Recognise and classify familiar two-dimensional shapes and three-dimensional objects using obvious features

Elaboration: focusing on geometric features and describing shapes and objects using everyday words such as 'corners', 'edges', 'faces'

Year 2 Shape:

- Describe and draw two-dimensional shapes with and without digital technology

Elaboration: identifying features of circles, triangles and rectangles, kites and rhombuses, such as straight lines, curved lines and counting edges and corners

Year 4 Shape:

- Compare and describe two-dimensional shapes that result from combining and splitting common shapes, with and without the use of digital technology.

Elaboration: identifying common two-dimensional shapes that are part of a composite shape by re-creating it from these shapes.

Visualisation and Transformation

Visualisation is understanding imagined movements of 2D shapes and 3D objects, by creating and manipulating mental images. As students work on shape-moving activities like jigsaw puzzles they practise identifying shapes that fit into remaining gaps and how they need to be manipulated. At first, students will need to move the pieces physically and test whether or not they fit, but with experience students will be able to determine mentally the effects of turning (rotating) and flipping (reflecting) a shape and how to manipulate it into place.

Transformations involve changing a shape's position, orientation or size, but not the kind of shape it is. Students need to understand that turning, flipping or sliding a shape does not change the kind of shape it is, and the moved/transformed shape is the same as the original, just in a different position or pointing in a different direction. To recognise they are the same they need to look at attributes, such as the lengths of sides and the size of corners (angles).

Games can provide opportunities for developing spatial abilities especially where visualisation is required. Such games also facilitate the development of both geometric and locational language.

Some games that encourage visualisation and language include:

- *Barrier game.* One student draws a shape on grid paper, or builds an object with cubes, then partner (on other side of barrier) replicates this by following oral directions.
- *Symmetry activities.* Complete the other half of a spatial picture or pattern using Pattern Blocks or Geoboards and elastics.
- *Tangram activities.* Use Tangram pieces to copy a given Tangram picture with or without internal lines around individual shapes shown.
- *Pentominoes game.* Use grid paper to draw and cut out pentominoes. On grid paper draw an 8 x 8 table. Players place pentominoes on the grid; the last person to place a pentomino on the board wins.

Language that students use to describe what they have done during spatial tasks will reveal their level of understanding more than the representations they create.

Australian Curriculum: Mathematics

The required content in transformation and visualisation within the Australian Curriculum is found in the Measurement and Geometry Strand, with further requirements in the Proficiency Strand. Students should be able to physically turn, flip and slide shapes (transform) and predict results by visualising. As they progress, success depends on students rotating mental images, then imagining the object rotating and being able to imagine the view from the other side. The ability to mentally rotate objects develops throughout primary and junior secondary years and is generally more difficult than mentally transforming shapes.

Measurement and Geometry
Year 2 Location and Transformation:

- Investigate the effect of one-step slides and flips with and without digital technologies
- Identify and describe half and quarter turns

Proficiency - Problem Solving:

Year 2 includes formulating problems from authentic situations, making models ..., matching transformations with their original shape ...

Year 5 Location and Transformation:

- Describe translations, reflections, and rotations of two-dimensional shapes. Identify line and rotational symmetries

Elaboration: identify the effects of transformations by manually flipping, sliding and turning two-dimensional shapes and by using digital technologies

- Apply enlargement transformation to familiar two-dimensional shapes and explore properties of resulting images compared with the original.

Year 5 Geometric Reasoning:

- Estimate, measure and compare angles using degrees. Construct angles using a protractor.

Proficiency - Understanding:

Year 5 ... describing transformations and identifying line and rotational symmetry

Australian Curriculum, General Capabilities

Numeracy - Using Spatial Reasoning

This element of numeracy involves students making sense of the space around them, and applying the skills of spatial reasoning by creating and interpreting maps using coordinates, using graphic organisers such as mind maps, conceptualising extremely small or extremely large spaces and using properties of shapes and objects in design and architecture. Detail can be seen in Figure I (Years 2 to 10).

Using spatial reasoning

By the end of Year 2 students:	By the end of Year 4 students:	By the end of Year 6 students:	By the end of Year 8 students:	By the end of Year 10 students:
recognise, visualise and classify familiar two-dimensional shapes and three-dimensional objects in the world around them describe position and movement in familiar contexts	identify and compare two-dimensional shapes and three-dimensional objects recognise symmetry in natural and built environments, and the importance of angles in symmetry	describe features of prisms and pyramids estimate, measure and compare angles using degrees	analyse the combination of different shapes and objects and their positions in the environment, in architecture, art and design	use their knowledge of right-angled triangles to solve problems involving direction and angles of elevation and depression
English - understanding and using the language of shape, position and movement Science - describing the shape of objects and the ways they move	English - using features such as shape and angle when creating visual texts Science: observing symmetry as a property of some living things History - building a 3-D structure of a past building	English - identifying how camera angles impact on the viewer's experience Science: explaining why some angles are used more frequently in built environments than others	English - understanding and using technical elements including shape, size, angle and framing to enhance meaning in visual and multimodal texts Science - describing the movements of objects using speed and direction	English - understanding and evaluating the effect of technical elements in visual texts
give and follow directions to familiar locations	show and describe position and pathways on grid maps	describe routes using landmarks and directional language such as north, south, east, west, north-west History - using maps to explain routes followed by explorers or patterns of development in the Australian colonies		

Figure 1. The scope of 'Using Spatial Reasoning' in the Australian Curriculum.

Part of this development is aiming at formal mathematics outcomes, such as Geometry, but spatial ideas and concepts can also be developed in the Arts, Geography, Science and other disciplines. The National Council of Teachers of Mathematics (2000) commented:

As students become familiar with shape, structure, location, and transformations and as they develop spatial reasoning, they lay the foundation for understanding not only their spatial world but also topics in mathematics and in art, science, and social studies. (p. 97)

The Mathematics Continuum

The Mathematics Continuum is a resource, available to all, on the Department of Education and Early Childhood Development (DEECD) website. The Continuum is organised into units or work, Foundation to Year 10, that provide evidence-based indicators of progress, teaching strategies and activities illustrative of these.

Indicators of progress are points on the learning continuum that highlight understandings required in order to progress, providing insight into mathematical content as well as teaching strategies to enhance teachers' content and pedagogical knowledge.

There are many resources teachers can and do use, with the Continuum offering excellent examples of tasks. This is available on the DEECD website at: http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/default.htm

Conclusion

Many studies show that students with specific spatial abilities are more competent at mathematics (Clements & Battista, 1992). The Australian Curriculum has recognised this by its emphasis on both geometry and spatial reasoning. Classroom programs need to reflect this in the tasks planned for students.

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FURTHERING THE USE OF *MATHEMATICA* AS A CAS TOOL

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For approved schools only, students enrolled in Further Mathematics in 2014 will be permitted to use CAS software for examinations in these studies where the use of technology is permitted. The following discussion considers how the CAS software Mathematica could be used as a teaching and learning tool in Further Mathematics and to assist in answering examination questions. The 2012 examinations will be used as a source of illustrative examples.

Introduction

There is a range of possible CAS software that could be used, including TI NSpire CAS and the CASIO ClassPad Manager. Why would one choose to use Mathematica – some of my reasons are:

- Mathematica is an extremely powerful teaching tool.
- It provides students with a tool for future studies.
- It has a HUGE support base on-line.
- It is cheap. It is free to all government secondary school students and individual students in non-government schools can purchase it for US$139.95 on line (US$69.99 for a year). Compare this with Office Works prices for TI-Nspire ($209) and CasioFX9860 ($219).
- It has applications in many studies in addition to mathematics.
- It is an excellent communication tools. It is designed for document development incorporating mathematical symbols and graphs and automatically provides power point versions. It can be saved in a variety of formats such as La Tex

(designed for publication), pdf and web format.

- It can be used to exchange interactive files between students and teachers.
- Its use in the current Mathematical Methods (CAS) examination trial proves it is an excellent medium for even the most sophisticated assessment.

For this paper, worked solutions to some questions from both 2012 examinations in Further Mathematics using *Mathematica*. (HREF2) are provided. It is important to note that there is relatively little need for calculators of any kind in these examinations. Here is a summary of the extent to which a calculator or CAS is needed:

Table 1

Need for calculator or CAS use in VCE 2012 Further Mathematics examinations based on number of marks.

Examination 1			
	No calculator	Scientific calculator	Cas
Core	5	8	0
Module1	2	7	0
Module2	4	5	0
Module3	4	3	2
Module4	2	6	1
Module5	9	0	0
Module6	6	0	3
Examination 2			
	No calculator	Scientific calculator	Cas
Core	6	6	3
Module1	5	6	4
Module2	7	3	5
Module3	3	7	5
Module4	0	11	4
Module5	15	0	0
Module6	7	0	8

In Examination 1, half the questions required no calculator use and 10% were most easily solved using CAS. For the remaining questions either CAS or a scientific calculator could be used.

In Examination 2, 40% of the questions required no calculator use and 25% were most easily solved using CAS. For the remaining questions either CAS or a scientific calculator could be used.

The usefulness of CAS in the examination varied significantly between modules. In Module 5 no question required use of a calculator or CAS.

The CAS applications are routine in most modules. The most challenging and rewarding applications are in Module 1: Number patterns (Arithmetic and geometric sequences and difference equations) and Module 4: Business-related mathematics (Loans and investments).

In the following excerpts from the Study Design (HREF3), the areas where *Mathematica* can be of most use are underlined.

Module 1: Number Patterns

Arithmetic and geometric sequences, including:

- the recognition of arithmetic sequences and the evaluation of terms and the sum of a finite number of terms and applications;
- the recognition of geometric sequences, the evaluation of terms and the sum of a finite number of terms; applications, including growth models;
- infinite geometric sequences and the sum of an infinite geometric sequence;
- contrasting arithmetic and geometric sequences through the use of graphs involving a discrete variable.

Difference equations, including:

- generation of the terms of a sequence from a difference equation, graphical representation of such a sequence and interpretation of the graph of the sequence;
- arithmetic and geometric sequences as specific cases of first-order linear difference equations;
- other first-order linear difference equations used to model change;
- setting up and using difference equations to represent practical situations such as growth models in various contexts (numerical and graphical solution of related equations);
- fibonacci and related sequences and applications (numerical and graphical solution of related equations).

Module 4: Business-Related Mathematics

Loans and investments, including:

- use and comparison of simple and compound interest in investment and loan applications without periodic payments;
- annuity investments involving a series of regular and equal deposits; consideration of the effects of initial and periodic deposit values, frequency of deposits, interest rate, and length of investment;
- the ordinary perpetuity as a series of regular payments from an investment that continues indefinitely;
- reducing balance loans as particular applications of annuities, with applications including housing loans, time payment plans (hire purchase) and credit/store cards; consideration of the effects of varying the repayment amount, the frequency of repayments, and the interest rate on the total repayment time and total interest paid;
- determination of effective interest rates from nominal interest rates.

Ways in which *Mathematica* can assist in these areas are illustrated below.

Difference Equations

Difference equations form the basis for arithmetic and geometric progressions, annuities and much more. *Mathematica* provides simple specialised tools for dealing with them.

Linear, first order difference equations are the simplest to explore. These are of the form t(n+1) = a t(n) + b

The simplest form is of the type t(n+1) = t(n) + 2. t(1) = 5.

This means 'new term' = 'old term' +2. First term is 5.
So t(1) = 5. t(2) = 5 + 2 = 7, t(3) = 7 + 2 = 9 and so on.
The terms will be 5, 7, 9, 11, 13, 15, ...
There is a common difference between successive terms of course.

```
Print[7 - 5, " ", 9 - 7, " ", 11 - 9]

2 2 2
```

This property establishes that the sequence or progression is arithmetic.

These are simple series where most of the required calculations can be done without sophisticated syntax. However, Mathematica has built in functions which provide identical procedures regardless of how complex the series become.

NestList

NestList provides a table of values for a sequence. It needs to operate on a user defined input. NestList uses a similar recursive definition to a difference equation. Consider the sequence 5, 7, 9, 11... A term is created by adding two to the previous term.

Hence we use:

```
arithmetic2[n_] := n + 2
```

The name is arbitrary but it is useful to make it meaningful. This example is an arithmetic sequence with common difference of 2.

```
NestList[arithmetic2, 5, 10]

{5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25}
```

The syntax looks a little strange but the rules are simple. First comes the variable name, next the starting term and then the number of applications of +2 (not the number of terms!)

Let's look at the sequence where we subtract 6, the first term is -8 and we want 5 terms.

```
arithmetic6[n_] := n - 6
```

It is better if the variable name is different - the 6 makes it different and reminds us it is connected with -6.

```
NestList[arithmetic6, -8, 4]

{-8, -14, -20, -26, -32}
```

Note to get 5 terms we need 4 applications of -6. What if we start with 'start'?

```
NestList[arithmetic6, start, 4]
```

```
{start, -6 + start, -12 + start, -18 + start, -24 + start}
```

If we have a common ratio of 2 between terms the sequence is geometric so we use:

```
geometric2[n_] := 2 * n
```

```
NestList[geometric2, 3, 10]
```

```
{3, 6, 12, 24, 48, 96, 192, 384, 768, 1536, 3072}
```

```
r = -4;
geometricr[n_] := r * n;
NestList[geometricr, 3, 10]
```

```
{3, -12, 48, -192, 768, -3072,
 12288, -49152, 196608, -786432, 3145728}
```

These sequences can be graphed and the effect of varying *a* or *r* can be noted.

```
r = 3;
a = 10;
geometricr[n_] := r * n; ListPlot[NestList[geometricr, a, 5]
 , PlotMarkers → Automatic]
```

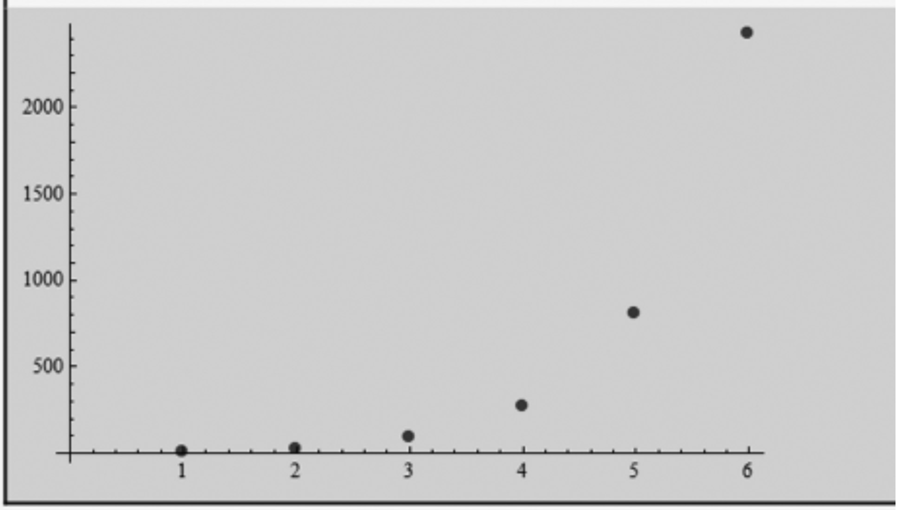


Figure 1. Graph of data where first term is a and common ratio is *r* .

Let's look at the sequence where we multiply by 1.065, the first term is 10000 and we want 5 terms. This is what happens if $10000 is invested at 6.5% interest paid annually for 5 years.

```
geometric65[n_] := 1.065 n

NestList[geometric65, 10 000, 5]

{10 000, 10 650., 11 342.3, 12 079.5, 12 864.7, 13 700.9}
```

If the interest compounded each month we would apply 6.5/12 % each month.

```
geometricmonthly[n_] := (1 + 6.5/(12 * 100)) n
```

```
NestList[geometricmonthly, 10000, 12]
```

```
{10000, 10054.2, 10108.6, 10163.4,
 10218.4, 10273.8, 10329.4, 10385.4,
 10441.6, 10498.2, 10555.1, 10612.2, 10669.7}
```

Note that at the end of the year we have $10669.70 compared to $10650 if the interest is applied annually.

Nest

This command is used if you only want one term. If you want the 123rd term of our initial sequence use:

```
Nest[arithmetic2, 5, 122]
```

```
249
```

Notice we use 122 because the 123rd term requires 122 applications.

RSolve

Use this if you want to find an imperical equivalent for a difference equation. For our first example:

```
RSolve[{t[n + 1] == t[n] + 2, t[1] == 5}, t[n], n]
```

```
{{t[n] → 3 + 2 n}}
```

The result is an imperical equation not a recurrence relation. Once again you can simple copy this from a resource file. In the syntax we use a double equals (= =) in all Solve commands; the difference equation and the boundary condition are bracketed together ,{ };

square brackets are used for defining t[n+1], t[n] and t[1] and we want to get t[n] in terms of n. Once we have this expression we can define t1[n_] as a function and evaluate it for any n. We use t1 rather than t because we may wish to keep t as an undefined variable for later work.

```
t1[n_] := 2 n + 3
```

FindSequenceFunction

This can also be used to define an empirical rule if the sequence of terms is known.

```
FindSequenceFunction[{5, 7, 9, 11, 13}, n]

3 + 2 n
```

This can be used as an alternative method for finding the 123rd term.

```
t1[123]

249
```

It can also be used to generate a table of values.

```
Table[2 n + 3, {n, 10}]

{5, 7, 9, 11, 13, 15, 17, 19, 21, 23}
```

Note the 10 here is the number of terms.

```
Table[{n, 2 n + 3}, {n, 10}]

{{1, 5}, {2, 7}, {3, 9}, {4, 11}, {5, 13}, {6, 15}, {7, 17}, {8, 19}, {9, 21}, {10, 23}}
```

```
Table[{n, 2 n + 3}, {n, 10}] // TableForm

1    5
2    7
3    9
4    11
5    13
6    15
7    17
8    19
9    21
10   23
```

We can sum the first 10 listed terms.

```
Total[{5, 7, 9, 11, 13, 15, 17, 19, 21, 23}]

140
```

ListPlot

This table can be plotted using ListPlot.

Note that in order to produce the clearest graph the axes origin is frequently not shown. We can fix this!

```
ListPlot[{5, 7, 9, 11, 13, 15, 17, 19, 21, 23}, AxesOrigin → {0, 0}]
```

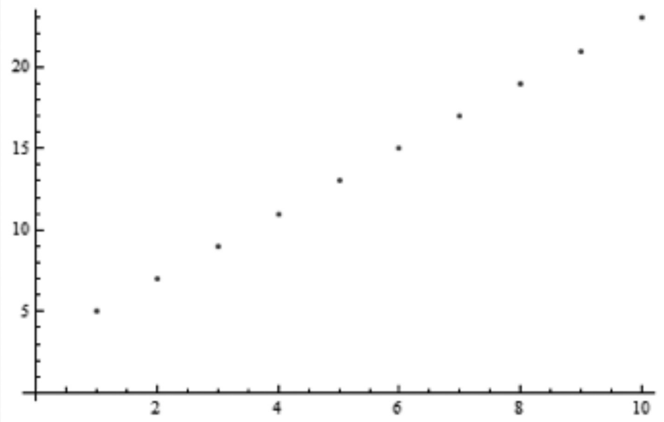


Figure 2. Adjusted ListPlot for axes origin.

If you want to you can cut-and-paste the sequence directly into ListPlot.

Sum

```
Sum[2 n + 3, {n, 10}]
140
```

This command will sum any sequence. It requires the empirical rule. This sums the first 10 terms. If you want to sum terms 20 to 37 use:

```
Sum[2 n + 3, {n, 20, 37}]
1080
```

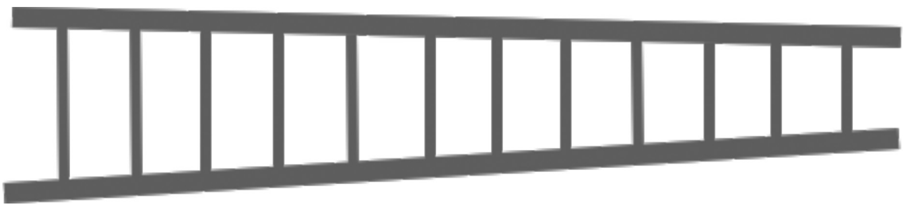
You can sum an infinite number of terms using Sum[2n+3,{n,1,Infinity}]. This is really only useful for geometric series where $|r| < 1$.

```
Sum[3*10^-n, {n, 1, Infinity}]
```

$\frac{1}{3}$

Context

Let's put some of this into context.



A ladder is to be constructed using aluminium tubing for the rungs. It tapers towards the top and has 12 rungs. The bottom rung is 40 cm long and the top rung is 35 cm long.

How much aluminium tubing is needed?

How long is each rung?

Let d be the unknown difference between successive rungs.

```
RSolve[{t[n+1] == t[n] - d, t[1] == 40}, t[n], n]
```

```
{{t[n] → 40 + d - d n}}
```

```
t2[n_] = 40 + d - d n
```

```
40 + d - d n
```

We can now use this to solve for d. If n = 12, t2(n) = t2(12)=35

```
Solve[t2[12] == 35, d] // N
```

```
{{d → 0.454545}}
```

We need to carry out some calculations where d is a constant. It is always safest to use a different name for these constants so we shall use d1.

```
d1 = 0.454545

0.454545
```

```
d1 * (-n) + d1 + 40

40.4545 - 0.454545 n
```

We need to sum the first 12 terms in this sequence to see what length of aluminium is needed.

```
Sum[40.454545` - 0.454545` n, {n, 12}]

450.
```

```
rungs[n_] := n - d1
```

```
NestList[rungs, 40, 11]
```

```
{40, 39.5454, 39.09090,
 38.6363, 38.1818, 37.7272,
 37.2727, 36.8181, 36.3636,
 35.9090, 35.4545, 35.0000}
```

This provides a check on the solution as after 11 iterations the value should be 35 and it is.

```
ListPlot[{40, 39.545455`, 39.090909999999994`,
  38.63636499999999`, 38.18181999999999`,
  37.727274999999985`, 37.27272999999998`,
  36.81818499999998`, 36.363639999999975`,
  35.90909499999997`, 35.45454999999997`,
  35.000004999999966`}]
```

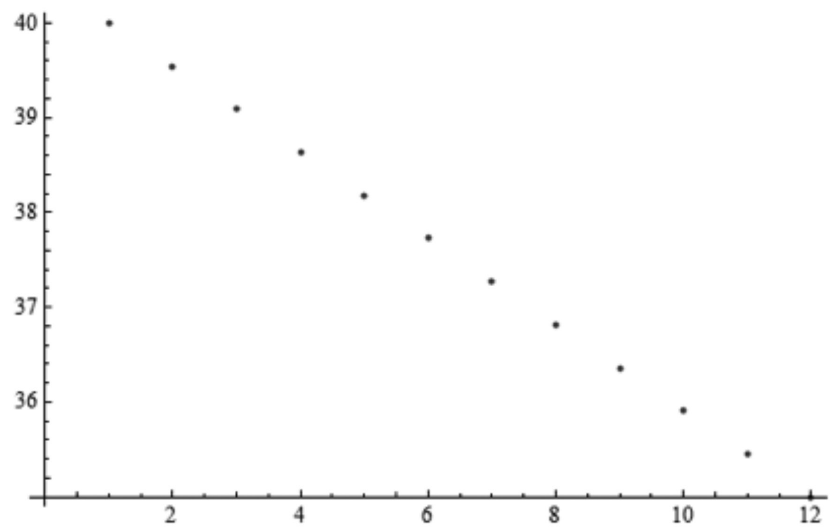


Figure 3. Graph showing progression of rung sizes on a ladder.

Annuity, Loan, Mortgage and Perpetuity

NetList

In a simple mortgage transaction interest may be calculated monthly and there may be monthly repayments. Consider the case where the annual rate is 7%, the initial mortgage value is \$200000 and monthly repayments are \$1500. The monthly interest rate is 7/12 % so this can be represented by the difference table t(n+1) =(1+7/1200)× t(n) - 1500, t(0)=200000.

n is the value of the mortgage at the start of a month; mortgage1[n_] is the value after interest has been calculated and a payment made.

```
mortgage1[n_] := (1 + 7 / 1200) * n - 1500
```

```
NestList[mortgage1, 200000, 12] // N
```

```
{200000., 199667., 199331., 198994., 198655., 198314., 197971., 1
```

You can see from this the mortgage is being reduced by paying 1500 per month. Trial and error can be used to find a 'break even' repayment where the mortgage doesn't change.

```
Manipulate[{mortgage1[n_] := (1 + 7 / 1200) * n - b;
  NestList[mortgage1, 200000, 12] // N}, {b, 1000, 2000}]
```


```
{{200000., 199512., 199020., 198526.,
  198030., 197530., 197027., 196521., 196013.,
  195501., 194986., 194469., 193948.}}
```

TimeValue

The TimeValue function serves the double purpose of calculating either present or future value. All that is required is a change of parameters.

Future value after 4 years of 5% on $1000 can be found from:

```
TimeValue[1000, 0.05, 4]
```

```
1215.51
```

To calculate past values just replace the 4 with -4 to work backwards.

```
TimeValue[1215.51, 0.05, -4]
```

```
1000.
```

Annuity

TimeValue is often used in conjunction with Annuity.

Consider our initial mortgage problem. 'In a simple mortgage transaction interest may be calculated monthly and there may be monthly repayments. Consider the case where the annual rate is 7% the initial mortgage value is $200000 and monthly repayments are $1500.'

The number of years required to amortise this loan equals 'time' which can be found by solving the following:

```
Solve[
 TimeValue[Annuity[1500, time, 1/12],
   EffectiveInterest[.07, 1/12], 0] == 200 000, time]
```

```
{{time → 21.5494}}
```

2012 Examination Questions

Some samples of how *Mathematica* syntax is applied in Further Mathematics examinations are given below.

Examination 1

Question 5

On the first day of a fundraising program, three boys had their heads shaved.

On the second day, each of those three boys shaved the heads of three other boys.

On the third day, each of the boys who was shaved on the second day shaved the heads of three other boys.

The head-shaving continued in this pattern for seven days.

The total number of boys who had their heads shaved in this fundraising activity was

A. 2187

B. 2188

C. 3279

D. 6558

E. 6561

There is no need to memorise or refer to the rule for summing a geometric progression.

```
Sum[3^n, {n, 1, 7}]

3279
```

Question 7

A dragster is travelling at a speed of 100 km/h.

It increases its speed by

- 50 km/h in the 1st second
- 30 km/h in the 2nd second
- 18 km/h in the 3rd second

and so on in this pattern.

Correct to the nearest whole number, the greatest speed, in km/h, that the dragster will reach is

A. 125

B. 200

C. 220

D. 225

E. 250

```
Sum[50 * 0.6^(n-1), {n, 1, Infinity}]
```

```
125.
```

Question 8

$15 000 is invested for 12 months.

For the first six months the interest rate is 6.1% per annum compounding monthly.

After six months the interest rate increases to 6.25% per annum compounding monthly.

The total interest earned by this investment over 12 months is closest to

A. $926
B. $935
C. $941
D. $953
E. $965

```
TimeValue[15000, EffectiveInterest[0.061, 1/12], 0.5]
```

```
15463.4
```

```
TimeValue[15463.353619466345`, EffectiveInterest[0.0625, 1/12], 0.5]
```

```
15952.9
```

Examination 2

Question 3

An area of the club needs to be refurbished.

$40 000 is borrowed at an interest rate of 7.8% per annum.

Interest on the unpaid balance is charged to the loan account monthly.

Suppose the $40 000 loan is to be fully repaid in equal monthly instalments over five years.

a. Determine the monthly payment, correct to the nearest cent.

Find the monthly payments on a $40000 mortgage amortized over 5 years at 7.8% nominal interest:

```
Solve[TimeValue[Annuity[payment, 5, 1/12],
  EffectiveInterest[.078, 1/12], 0] = 40000, payment]

{{payment → 807.233}}
```

c. Arthur's wife, Martha, invested a sum of money at an interest rate of 9.4% per annum, compounding quarterly.
She will be paid $1260 per quarter from her investment.
After ten years, the balance of Martha's investment will have reduced to $7000.
Determine the initial sum of money Martha invested.
Write your answer, correct to the nearest dollar.

```
Solve[TimeValue[Annuity[1260, 10, 1/4], EffectiveInterest[.094, 1/4], 0] == amount - 7000, amount]

{{amount → 39443.7}}
```

Conclusion

In this paper I have identified areas of the Further Mathematics course where CAS can be particularly valuable both as a teaching tool and to assist student in completion of assessment tasks. The relevant *Mathematica* syntax has been presented and its application to a sample of examination questions given.

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MODELLING DATA WITH *MATHEMATICA*

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This article explores two reasons why many students meet a brick wall when introduced to algebra in the middle years: the dominance of x as a universal variable and the absence of realistic contexts. One day x can be length, next width and then area! Mathematica works comfortably with length, width and area as variable names. It also facilitates the manipulation of 'dirty' data typically associated with genuine modelling without compromise. The article also illustrates an approach to curriculum development using an introduction to Algebra as an example.

Introduction

Many students experience a failure to cope with algebra – why is this? Many answers have been proposed but this paper directs attention to two:

Reason One: x

In most text books x is *the* pro-numeral introduced and dominates exercises. To cope, students have to treat x as a chameleon in that it can represent any quantity – length, time, height, area. Why can't a set of length values have an element called *length*? Why can't *area* = *length* × *width*?

Eventually we need to include x and y because

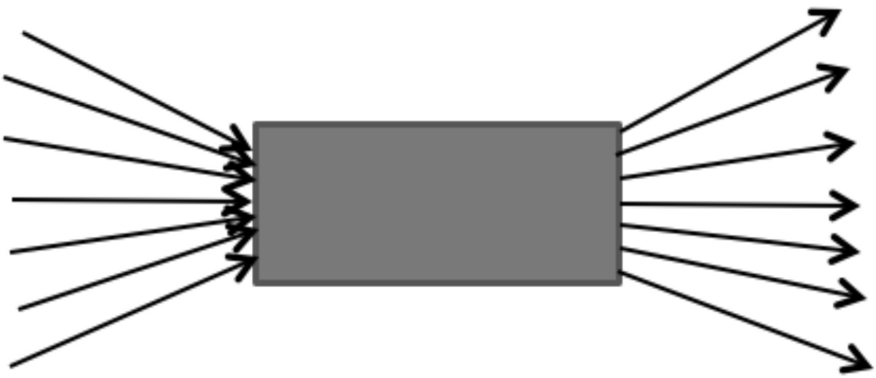
- Information technology often defaults to give output in terms of x and y
- Examinations are dominated by x and y as variable names

Reason Two: Absence of Context

A principal long term objective of teaching Mathematics is its application in solving real problems, hence we need to introduce the skills and processes associated with algebra through real contexts. This provides the 'advanced organizers' students need for future applications.

The Hook - Activity One: Calendar Maths

Students have a right to be interested in what teachers present them. Some curriculum models suggest a structure consistent with this simple image:



Problems Skills and procedures Applications

The introductory problems need to be rich and approachable by all students. It should come from their experience but does no need to have a practical focus. After all, much of a student's life is focussed on games of one sort or another. Calendar Maths fits the bill.

This is a powerful lesson focus.

- The data are real and readily accessible.
- Domain and range issues are practical
- By changing the shape and the number of cells covered different linear situations can be modelled. The constant multiplier is related to the number of cells.
- Equations can be established in a variety of ways and can be solved and the solutions validated
- Equations without solution can be discussed
- The variable name is not x

December

Mon	Tue	Wed	Thur	Fri	Sat	Sun
						1
2	3		5	6	7	8
		11		13		15
16	17		19	20	21	22
23	24		26	27	28	29
30						

Select a student with a birthday in December and have them move the circle over their birthday and move the 4 x 4 square to cover a square of numbers including the birthday. As soon as the 4 x 4 square is drawn you know the final result of the process to follow will be 56 (in this case) and the class should be asked to remember that number. Then the student covers every other date in the same row and column as their birthdate with a blue square. A second number is circled from those remaining and the other dates in the same row and column as that are covered. Continue until there are no remaining dates inside the 4 x 4 square and there will be four dates circled. The total is added and as if by magic the total is 56! The task can be repeated and regardless of where we start the total is still 56. WHY IS IT SO? The mathematics of algebra which we are about to study holds the key.

Let's start with a simpler but related problem. In the table below the L shape (standard shape, no rotations allowed!) should have the *date* = 4 at the top. The *total* = 27. Note: no *x*. *date* and *total* are sensible variable names.

Mon	Tue	Wed	Thur	Fri	Sat	Sun
						1
2	3	4	5	6	7	8
9	10	11	12	13		15
16	17	18	19	20	21	22

Start by gathering a set of data. Electronic versions of the figures used will be provided in the workshop. With these the L shape can be moved over the table.

Complete this set of data:

date	*total*
3	
4	27
5	
6	
7	

It has been suggested that the total for the L Shape can be found by multiplying the date by 3 and adding 15. Does this work for your results?

We can write this calculation as *total* = 3 × *date* + 15 or as

total = 3 *date* + 15. Use this rule to predict the *total* if *date* = 10.

Can you find a location for your L so that *total* = 66 just by trial and error?

Can you find a location so that *total* = 61?

Questions like these can be explored using the *Mathematica* Solve function. If *total* = 66 we can solve for *date*.

```
Solve[66 == 3 date + 15, date]
{{date → 17}}
```

This will be the *date* if *total* = 66.

If *total* = 61 we get *date* = $46/3$ which is not a whole number so there is no *date* if *total* = 61.

```
Solve[61 == 3 date + 15, date]
{{date → 46/3}}
```

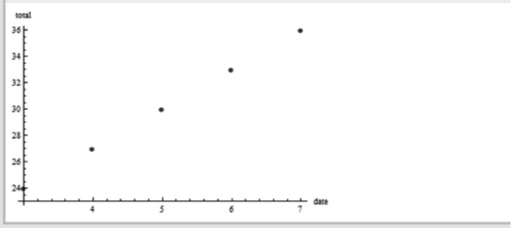
What is the smallest value *date* can have? What is the largest value *date* can have? What values can't *date* have?

By the way, are you getting a bit sick of writing *date* and *total*? If you like you can shorten *date* to *d* and *total* to *t*. *d* and *t* are called pro-numerals because they are single letters which could take any of the allowed values.

We can enter the data table into Mathematica and look at a graph of how the variables are related. This is in response to ACMNA178-7-0 Given coordinates, plot points on the Cartesian plane, and find coordinates for a given point.

```
calendardataL = {{3, 24}, {4, 27}, {5, 30}, {6, 33}, {7, 36}}
{{3, 24}, {4, 27}, {5, 30}, {6, 33}, {7, 36}}
```

```
ListPlot[calendardataL, AxesLabel → {"date", "total"}, PlotMarkers → Automatic]
```

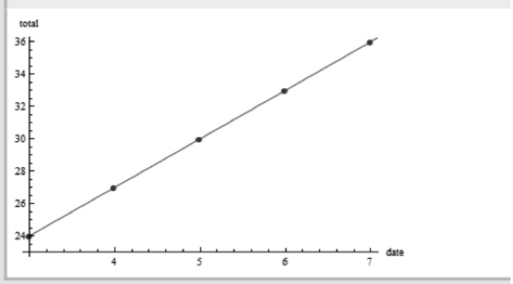


It looks as though the points are in line. Mathematica will tell us the rule for that line. The parameters are real numbers so 3 is written as 3• and 15 as 15• 3. *date* is the same as 3 × *date*.

```
line = Fit[calendardataL , {1, date}, date]
15. + 3. date
```

Mathematica says that *total* = 15 + 3 × *date*. Is this the same as *total* = 3 *date* + 15? We can show that this equation matches the lines by plotting it on top of the data points.

```
Show[ListPlot[calendardataL, AxesLabel → {"date", "total"},
  PlotMarkers → Automatic], Plot[15 + 3 date, {date, 3, 8}]]
```



For any L Shape write a rule for each of the bottom two cells in terms of *date*.

date	
***date* + …**	***date* + …**

Write down the value of *total* in terms of the sum of the three cells. Cross off all the values in this table that *date* cannot have.

						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30						

The set of numbers not crossed out is called the domain.

What is the minimum value *total* can have? What is its maximum value? What values can't *total* have?

Describe the pattern of allowable numbers. This set of numbers is called the range.

<table>
<tr>
<td>Gather a second set of data, this time with an I shape:
<table><tr><td>6</td></tr><tr><td>13</td></tr></table>Use dateI for the input and totalI for the total.</td>
<td>A third set can be found with a box:
<table><tr><td>6</td><td>7</td></tr><tr><td>13</td><td>14</td></tr></table>Use dateB for the input and totalB for the total.</td>
</tr>
</table>

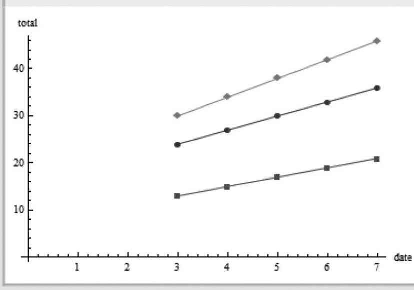
Here are some sample data sets:

```
calendardataI = {{3, 13}, {4, 15}, {5, 17}, {6, 19}, {7, 21}}
{{3, 13}, {4, 15}, {5, 17}, {6, 19}, {7, 21}}

calendardataB = {{3, 30}, {4, 34}, {5, 38}, {6, 42}, {7, 46}}
{{3, 30}, {4, 34}, {5, 38}, {6, 42}, {7, 46}}
```

When all three data sets are graphed we get:

```
ListLinePlot[{calendardataL, calendardataI, calendardataB},
 AxesLabel → {"date", "total"}, PlotMarkers → Automatic,
 AxesOrigin → {0, 0}]
```



This set of graphs provides a good basis for discussion. Why are the graphs all linear? Why are the gradients different? If I create a new shape which includes three dates how will it's graph relate to that for the L shape?

How Mathematics Should Look to Students

The value in working with meaningful contexts was explained by Conrad Wolfram, the strategic director of Wolfram Research, the technology company founded by his older brother Stephen, who developed Mathematica and Wolfram Alpha, the answer engine. At a 2010 TED (Technology, Entertainment, Design) talk (HREF1), he argued that teachers "were wasting time forcing students to do 'by hand' processes that they could accomplish faster and more accurately with computers or calculators. [He warned] of a growing chasm between the increasing complexity of modern life and a math curriculum where students apply procedures they don't understand for reasons they don't get".

He continued by saying that "in schools we've made everything so people can hand-calculate it — or if they're lucky, use a calculator. [This] means, the problems you can get people to solve are totally simplistic. Instead of 10,000 data points, which you might need in order to analyze in a real situation, you are given five and asked to find the mean, because that's all you can do by hand".

(In Mathematics) "we're talking about ... anything that is amenable to this four-step process:

- *You ask the right questions about the situation: "What is it that we're really trying to figure out?"* (Knowledge and comprehension: You need to identify the 'given' and the 'goal'. Determine what information is superfluous and what is needed but not provided)
- *You turn that into mathematics — an expression or a program: something you can compute and answer using calculating processes.* (Knowledge, comprehension, application, analysis)
- *The third step is calculating. That gives you an answer in a mathematical format.* (Knowledge, comprehension, application)
- *Finally you have to go back from mathematics to your original problem.* (Analysis and synthesis)"

The underlined elements above have been added by the author of this paper and refer to Bloom's levels. A fifth step should be added – validation of result, both as a correct Mathematical result and as a meaningful result in the initial context. This corresponds to Bloom's *Evaluation.*

The third step in this process invites sensible use of computers, both for computation and for assistance in the development of knowledge of processes and skills. But this is only 20% of the picture – steps one, two and four (and five) demand understanding and the ability to think mathematically.

Backwards Design of Curriculum

Backward design can be employed to assist in translating Conrad Wolfram's four step process for using Mathematics into a system for constructing a curriculum. It emphasizes the endpoint you wish to achieve and progresses through six steps. Coupled with responses to Bloom's levels it can lead to development of a sound curriculum.

Step one:	Deciding what is it you want to do? Identifying the big picture?
Step two:	Explaining why it is important?
Step three:	Determining which constraints are IMMOVEABLE?
Step four:	Making assessment decision?
Step five:	Deciding what tasks students will complete and which resources they will use in order to be prepared for the assessment?
Step six:	Designing student work sheets which ensure the curriculum decisions are translated directly and efficiently into student' tasks?

Here is an example of the first three steps in the backward design process for Year Seven Number and Algebra: Patterns and algebra and Linear and non-linear relationships.

Step One.

What is it you want to do? What is the big picture?

Year Seven is the starting point for a study of algebra. What is it we are trying to do when introducing algebra? We need to consider:

- Algebra as generalized arithmetic. Students are expected to create symbolic representations of real situations.
- Students need to gain confidence in using variable names to stand for any element in a set of numbers, not just one of them.
- How data can be represented and situations modelled:
  - Description of the situation
  - A set of data
    - Exact values
      - Discrete

            - Continuous
        - Measurements
            - Accuracy
            - Error
    - A graph. They need to plot point on the four quadrants of the Cartesian plane
    - An empirical rule
    - Summary statistics
        - Averages
        - Spread
        - Correlation
    - Interpolation and extrapolation
    - Estimation
    - Validation of results
    - Classification of relationships without being restricted to linear equations
        - Linear
        - Non-linear
        - Scattered
    - Solution of equations

A common feature of algebraic situations is that we are dealing with bivariate data – inputs and outputs.

A suitable big picture focus would be:

An introduction to investigation of data

Step Two:

Why is it important?

Demonstrates usefulness of algebra in real contexts

Background for future Mathematical learning:

- A lot of modelling involves collection of data followed by analysis. Most data at this level of education will consist of pairs of values for variables. Graphing a data set will aid interpretation. Finding an algebraic model may be possible and problem solving may require algebraic solution of equations.
- Distinguishing between linear and non-linear relationships
- Distinguishing between scattered and correlated data

- Linear relationships are a starting point for polynomial relationships
- Linear relationships are associated with producing the next term in a sequence by adding a constant; exponential relationships are associated with multiplying by a constant
- It is common to express relationships as an equation involving variables eg *area* = *length* × *width*

Step Three:

What constraints are IMMOVEABLE?

Australian Curriculum Content Descriptions and Elaborations (HREF2)

Patterns and Algebra: Create algebraic expressions and evaluate them by substituting a given value for each variable. Extend and apply the laws and properties of arithmetic to algebraic terms and expressions

Linear and non-linear relationships: Given coordinates, plot points on the Cartesian plane, and find coordinates for a given point. Solve simple linear equations. Investigate, interpret and analyse graphs from authentic data

Consideration needs to be given to pre-requisites and outcome from other parts of the curriculum including work on sequences; the Cartesian coordinate system; integers; formulas for area; fractions, decimals and percentages; data displays; corresponding, alternate and co-interior angles; conditions for two lines to be parallel; solution of simple numerical problems using reasoning and that the angle sum of a triangle is 180°.

Looking ahead we need to prepare students to solve problems involving rates and ratios; extend and apply the distributive law to the expansion of algebraic expressions; factorize algebraic expressions by identifying numerical factors; plot linear relationships on the Cartesian plane and solve linear equations using algebraic and graphical techniques.

There are also general capabilities, cross-curricular priorities and proficiency strands to consider. The following excerpt illustrates the emphasis placed on understanding which is the link to meaningful contexts.

Australian Curriculum Proficiency Strands (HREF3)

Understanding: Students *make connections* between related concepts and progressively *apply* the familiar to develop new ideas.

Fluency: Students are fluent when they calculate answers efficiently.

Problem Solving: Students solve problems when they use mathematics to represent *unfamiliar* or meaningful situations, when they *design* investigations and plan their approaches and when they apply their existing strategies to seek solutions.

Reasoning: Analysing, proving, evaluating, explaining, inferring, justifying and generalising.

Australian Curriculum General Capabilities (HREF4)

Numeracy: Students need to apply Mathematical understanding and skills in context, both in other learning areas and in real world contexts.

Information and communication technology (ICT) capability. ICT capability involves students in learning to *make the most of the technologies available to them, adapting to new ways of doing things as technologies evolve* and limiting the risks to themselves and others in a digital environment.

Critical and creative thinking: Students develop critical and creative thinking as they learn to generate and evaluate knowledge, ideas and possibilities, and use them when seeking solutions. Engaging students in reasoning and thinking about solutions to problems and the strategies needed to find these solutions are core parts of the Mathematics curriculum.

School mission statements also dictate roles for Mathematics and for mathematics teachers.

There are many other constraints such as parental expectations, performance on national and international tests, maintenance of adequate numbers in senior mathematics classes, available resources, cost and compatibility of a school's curriculum to the norm.

We shall now consider a set of activities which are consistent with the aim of using meaningful contexts whilst using sensible variable names assisted by appropriate technology.

Activity Two: Leaking Water (Almost Linear)

The focus is to investigate how long it takes for water to drain out of a bottle.

Prepare the bottle by cutting off the base and drilling a 4 mm hole in the cap. Mark the bottle in equal intervals (say 5 cm) from the top of the cap.

Fill the bottle with water and time how long it takes to reach each marker. Call the top marker 20. The others can be 15, 10, 5 and 0 (for the cap).

A stop-watch on a mobile 'phone can be useful for this. For a one person job, pressing the lap button each time a marker is reached can be very practical. Lap 1 is the time taken between markers 20 and 15; lap 2 is between markers 15 and 10 ... and so on.

```
drainingtime = {{20, 79.8}, { 15, 63.8}, { 10, 43}, { 5, 19}, {0, 0}}

{{20, 79.8}, {15, 63.8}, {10, 43}, {5, 19}, {0, 0}}
```

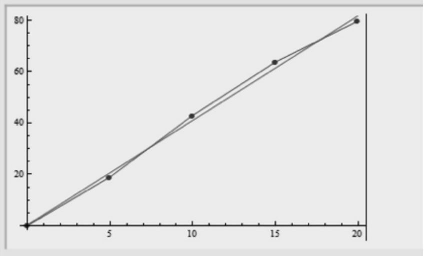
Will all the laps be of equal time?

What is the rule linking marker heights and the time taken to reach them?

```
line = Fit[drainingtime, {1, height}, height]

0.24 + 4.088 height
```

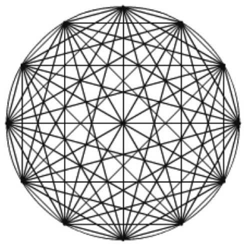
```
Show[ListLinePlot[drainingtime, PlotMarkers → Automatic],
 Plot[0.24 + 4.08 height, {height, 0, 20}]]
```



The following table identifies other activities which could be used. Student worksheets could be created for each one. It is not intended that students attempt all activities but that the results of all activities be presented as posters so that all students can compare the type of data resulting. Principally we want students to be able to discriminate between linear and non-linear relationships and understand that real data frequently results in a scatter graph.

Completion of worksheets based on these activities could enable students to address all outcome statements associated with Year Seven Number and Algebra: Patterns and algebra and Linear and non-linear relationships. The table at the end of this paper can be used to demonstrate this.

Activity three: Mystic rose



Twelve points are drawn around a circle. Each point is joined to every other point. How many lines are needed?
Two students argue about the answers, Sarah used 12+11+10+9+8+7+6+5+4+3+2 and George said the answer was $\frac{12\times11}{2}$. Who was right? How did they get their answers?

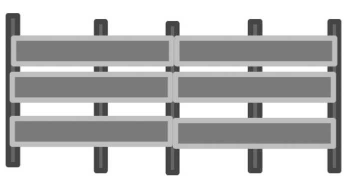
Activity four: Cost per item (Non linear)

In this case you have $12 to spend. If this buys you 1 item the item will have cost $12. Complete this table:

Number of items for $12	*Cost per item ($)*
1	12
2	6
3	4
4	3

Graph the data. Establish a rule for calculating cost per item from number of items.

Activity five: Fences



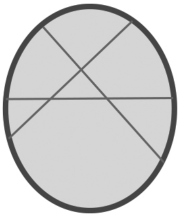
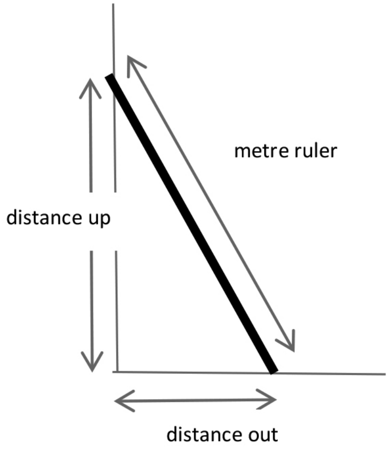
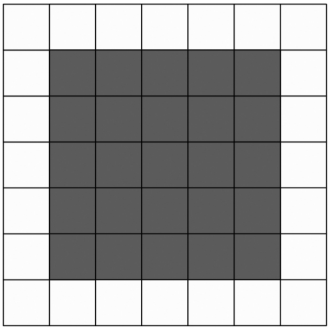
Investigate the relationship between the number of posts and the number of rails. The gradient of the graph depends on the number of rails per section so it leads to an investigation of what the gradient of a linear model depends on.

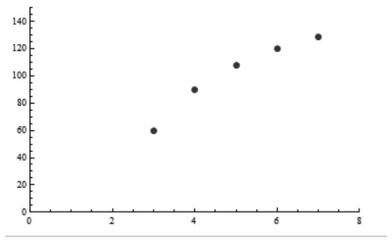
Activity six: Drop and rebound (Almost linear)

Drop a golf ball from various heights and record the bounce height. Videotape with a mobile phone adds interest and accuracy.

Activity seven: Perimeter and area (Linear and non linear)

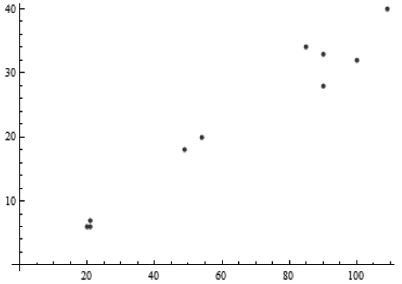
Activity eight: Cuts versus pieces

This is particularly interesting because the one context leads to useful linear and non-linear results	What is the maximum number of pieces a circle can be cut into with four cuts? What about ten cuts? 
Activity nine: Metre ruler against a wall (Non linear)  Record and plot distances up and out. Choose aspect ratio carefully and a circle results.	**Activity ten: Paving around a garden (Linear)**  How is the number of pavers related to the length of the garden bed?
Activity eleven: Vertex angle in a regular polygon (Non linear)	**Activity twelve: Length and width of leaves (Scatter)**



The data source is a sketch so no equipment is needed. The activity reinforces the angle sum of a triangle because each polygon is divided into the minimum number of triangle containing all vertex angles.

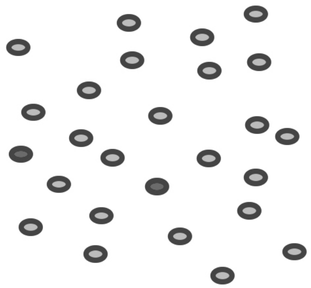
Select leaves of different sizes from one bush or tree.



Activity thirteen: Arm span and height (Scatter)

What is the relationship between arm span and height? With what accuracy can you predict height from arm span? Is there a gender factor?

Activity fourteen: Linking letters (Scatter)



Prepare a sheet which has a letter in the alphabet in each dot. Time how long it takes to join the letters in alphabetical order. Provide ten minutes 'study time' and retest.

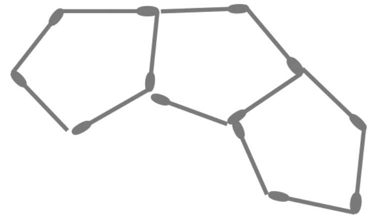
Activity fifteen: Tables speed

X	*1*	*6*	*10*	*4*	*8*

Activity sixteen: Match stick patterns

9					
3					
2					
7					
5					

Time how long it takes to carry out the 25 multiplications. Provide ten minutes ‘study time’ and retest.



What is the relationship between the number of pentagons and the number of matches? How do the graphs compare for triangles, quadrilaterals, pentagons and hexagons?

Recording Relationships

The worksheets need to illustrate how each activity relates to the content descriptions. This can be achieved by recording the relationship on a table such as this:

Learning outcomes and units of work	Battleships	Plotting Points	Calendar Mathematics	Perimeter and area	Cost per item given total	Leaking Water	Metre ruler against a wall	Paving around a garden bed	Angles in a regular polygon (sum and vertex)	Length and width of leaves	Arm span and height	Linking letters	Table speed
ACMNA175-7-0 Introduce the concept of													

variables as a way of representing numbers using letters													
ACMNA176-7-0 Create algebraic expressions and evaluate them by substituting a given value for each variable ACMNA177-7-0 Extend and apply the laws and properties of arithmetic to algebraic terms and expressions													
ACMNA178-7-0 Given coordinates, plot points on the Cartesian plane, and find coordinates for a given point													
ACMNA179-7-0 Solve simple linear equations													
ACMNA180-7-0 Investigate, interpret and analyse graphs from authentic data													

References

HREF1: TED. (2010). Retrieved September 4th 2013 from http://www.ted.com/talks/conrad_wolfram_teaching_kids_real_math_with_computers.html

HREF2: Australian Curriculum: Mathematics. (20013). Retrieved September 4th 2013 from http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10

HREF3: Australian Curriculum: Mathematics – Proficiency Strands (2013). Retrieved September 4th 2013 http://www.australiancurriculum.edu.au/Mathematics/Content-structure

HREF4: Australian Curriculum: Mathematics –General Capabilities. (2013). Retrieved September 4th 2013 from http://www.australiancurriculum.edu.au/Mathematics/General-capabilities

WORLD DATA + *MATHEMATICA* = AUSTRALIAN CURRICULUM

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The rationale for the Australian Curriculum: Mathematics includes a statement that "in geography, interpretation of data underpins the study of human populations and their physical environments" and that the curriculum "encourages teachers to help students become self-motivated, confident learners through inquiry and active participation". In this paper we will show you how to design efficient and motivating units focused on geographical and other geometrical aspects of Year Seven to Ten Mathematics which are consistent with this rationale and are related to the Mathematics of Planet Earth. Mathematica and Wolfram Alpha will be the IT tools used to facilitate this.

Introduction

On a recent Melbourne radio program a teacher telephoned in and said the equivalent of *"Good teachers need to find a hook for students. Even as a teacher I find the Year Nine program boring."*

The rationale for the Australian Curriculum: Mathematics (Australian Curriculum) (HREF1) formally supports this sentiment. We should not expect students to be able to transfer Mathematical skills and processes taught in the absence of a context into effective practical problem solving and modeling.

Which of these two samples of school curriculum would a student find more motivating:

Sample One	**Sample Two**
Chapter 1Numeracy	Topic 1 Turning back the boats
Chapter 2Number skills	Topic 2 The Earth in space
Chapter 3 Algebra	Topic 3 Humans and other species
Chapter 4Linear Equations	Topic 4 My perfect overseas holiday
Chapter 5 Congruence and similarity	Topic 5 Why should I be vaccinated?
Source: The first five chapter headings in a Year Nine Mathematics text published in 2012 for the Australian Curriculum	

The purpose of this article and the associated presentation assumes the answer to be Sample Two and provides a pathway towards creating these and other topics related to the Mathematics of Planet Earth

Building Curriculum by Backwards Design

Here we need to start our planning with the endpoint in mind. We want to have students understand the Mathematics of Planet Earth - that is the end point of their learning and the starting point for curriculum design.

The Australian Curriculum content descriptions and elaborations (HREF2) make very little specific reference to the relationship between Mathematics and Earth. We need a link between the skills and processes in the Australian Curriculum and a set of contexts underpinning our desired endpoint.

What leads are provided by the Australian Curriculum?

What do teachers think of when asked about the Mathematics of Planet Earth?

What would students list?

What does the scientific community include?

The Rationale for the Australian Curriculum

The rationale for the Australian Curriculum includes that "in geography, interpretation of data underpins the study of human populations and their physical environments". So we could include:

- Statistical analysis of population data for individual countries. Ranking countries on population size, gross domestic product, employment, literacy, land mass, religion, size of army and poverty.

- Ranking cities within countries and identifying capital cities
- Refugees and immigration

Students

Some students have suggested:

Mobile telephone coverage	Where are:	Historical changes of countries' boundaries
Social media penetration	Continents	Climate change
Travel plans and cost	Countries	Saving the planet
Turning back the boats	Cities	

The Scientific Community

The Mathematics of Planet Earth Australia 2013: Conference (MPE2013) in July in Melbourne (HREF3) focussed on the following themes:

A planet organized by humans	**A planet at risk**	**A planet to discover**
Conquering numerical error	Catastrophe modelling	Realizing our subsurface potential
Data science	Extreme weather and climate events	
Molecular bio-invasion	Climate variability and change	
Weather forecasting	Land – atmosphere exchanges	
Australian population census	Simulation and reproducibility in global climate modelling	
Measures of Australia's progress	Bio-invasion and biodiversity	
A data based view of our world		

The MPE2013 website organizes resources in categories which I have placed into possible theme clusters:

The Earth in Space – astronomy, solar systems,

Humans and other species – biology, biodiversity, biosphere, biogeochemistry, ecology, evolution, sustainability,

Things are heating up – climate change, atmosphere, carbon cycle, climate modelling, global warming, natural disasters, meteorology, paleoclimate, sea ice, weather

Why should I be vaccinated? – disease, disease modelling, epidemiology, natural disasters, public health

Facebook and other networks – finance, political systems, social systems, transport

My perfect overseas holiday – the physical planet, geophysics, ocean, atmosphere, renewable energy, data assimilation, data visualization

Turning back the boats – disasters, asylum seekers, migration

On, under and below the Earth – Map projections, latitude and longitude, surface area, length of the equator, distance between points on the Earth's surface (find shortest path using string on a globe) political boundaries, boundaries between countries (the four colour problem), atmospheric layers, Earth's composition, mineral wealth, oil

Our home and surrounding structures – Architecture, building plans, subdivision, provision for recreation

There are also public lectures available on the MPE website (HREF4) which support some of these themes but add information on a range of topics including plague, melting polar ice caps, the internet, epidemics, fracking, pandemics, crime, air pollution and tsunamis.

Curriculum materials (HREF5) are also provided on the MPE website on Networks (biodiversity, biology, disease, ecology, epidemiology, the ocean, public health, social systems and media, communications and transportation); Space Maths (atmosphere, climate, energy, global warming, meteorology, natural disasters and the ocean) and Mathematics of Remote Sensing (astronomy, atmosphere, climate, energy, global warming and the ocean). All this from one web site – finding resources is not the problem.

Developing Units

Unit headings or themes are clusters of related contexts. For an efficient distribution of content the themes are initially considered to be exhaustive (each context can be classified into one of them) and mutually exclusive (no context belongs in more than one). You may have a completely different set of themes in mind and that will be of little consequence – any workable set will do. The purpose is to relate the Mathematics being taught to a meaningful and inclusive set of themes.

Some issues remain.

- Which themes are relevant to students?
- Which contain information which is accessible and not too difficult?

Next the content descriptions for Years Seven to TenA (principally in Measurement and Geometry and Statistics and Probability) are linked to the contexts. These content descriptions and elaborations may prompt additions to the list of contexts.

The themes are thus used to organize the content descriptions. As far as possible each content description is related to only one theme. This ensures that the maximum range of content descriptions can be approached via the themes.

Here is how one theme might be developed:

Turning Back the Boats

Table 1 *A Link Between Australian Curriculum Content Descriptions and Related Contexts for 'Turning Back the Boats'*

Plot linear relationships on the Cartesian plane with and without the use of digital technologies ACMNA 193-8	Historic data and prediction – linear models. Given data to June how could you estimate the number for the calendar year? Detention - time in detention, location,
Sketch simple non-linear relations with and without the use of digital technologies ACMNA 296-9	The boats – number of boats, size, people per boat, proportion reporting distress
Explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials using digital technology as appropriate ACMNA 239-10	Historic data and prediction – exponential models.
Solve simple exponential equations ACMNA 270-A	Historic data and prediction – exponential models.
Use scatter plots to investigate and comment on relationships between two continuous variables ACMSP 251-10	Given data to June how could you estimate the number for the calendar year?

	Bivariate data.
Investigate and describe bivariate numerical data where the independent variable is time ACMSP 252-10	
Evaluate statistical reports in the media and other places by linking claims to displays, statistics and representative data ACMSP 253-10	Critical analysis of media reports.
Describe events using language of 'at least', exclusive 'or' (A or B but not both), inclusive 'or' (A or B or both) and 'and' ACMSP 205-8	Survey questions and their analysis.
Construct and compare a range of data displays including stem-and-leaf plots and dot plots ACMSP 170-7	Statistical data on 'irregular maritime arrivals' – nationality, age, gender, travel routes, time taken, speed, proportion granted asylum, health. Cost – cost per person for passage; cost to Australia. World refugee situation. Illegal immigrants in general.

The next step is to relate student tasks to the themes via the learning content descriptions. The principal tool we have used for the analysis of the data is *Mathematica*. In the following sample we have shown how a variety of *Mathematica* commands can be used in connection with outcome statements from Years Seven to Ten.

Construct and Compare a Range of Data Displays Including Stem-and-Leaf Plots and Dot Plots ACMSP170-7

The focus is on statistical data on 'irregular maritime arrivals' - nationality, age, gender, travel routes, time taken, speed, proportion granted asylum, health. It also deals with cost - cost per person for passage; cost to Australia. We can include the world refugee situation and illegal immigrants in general.

The tasks are to construct and compare. Students need to know which data displays to consider and need to know how to produce them with and without the aid of technology.

The roles of technology might be:

- Validation of student product

  Speed and accuracy for producing a relatively large number of displays from data which may be from extensive secondary sources involving 'dirty' numbers (very large, very small, involving fractions)

Question 1

What is the gender distribution of irregular maritime arrivals? Have the proportions changed over time?

Table 2 *Gender distribution of irregular maritime arrivals between 2008-9 and 2011-12. (HREF6)*

Sex	2008-08	2009-10	2010-11	2011-12
Female	7	140	330	752
Male	202	2011	2389	4014
Total	209	2151	2719	4766

This problem involves a calculation of percentages. The male percentage in the 2008-2009 financial year is given by:

```
malepercentage0809 = 202 * 100 / 209

20200
----- // N
 209

96.6507
```

Mathematica can deal with entire strings all at once. Just divide the string of male values by the string of totals and multiply by 100. The extension //N is needed to convert the exact fractional values into decimal approximations.

```
{{202, 2011, 2389, 4014} / {209, 2151, 2719, 4766}} * 100 // N

{{96.6507, 93.4914, 87.8632, 84.2216}}
```

This data can be simplified to two significant figures.

```
percentagemales = N[{{202, 2011, 2389, 4014} / {209, 2151, 2719, 4766}} * 100, 2]

{{97., 93., 88., 84.}}
```

A dot plot can be created for the four year trend.

```
ListPlot[percentagemales, AxesOrigin → {0, 0}, PlotRange → {0, 110},
 PlotStyle → PointSize[0.02]]
```

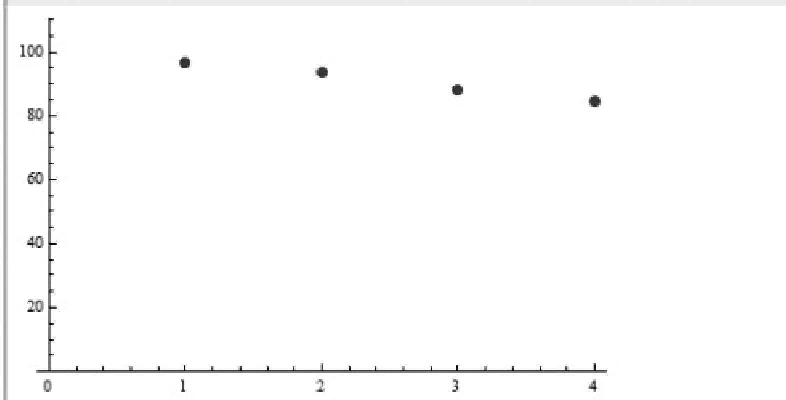

Figure 1. Gender distribution of irregular maritime arrivals between 2008-9 and 2011-12

Question 2

How has the age distribution changed since 2008 - 2009?

Age at Request	2008-09	2009-10	2010-11	2011-12
0-17 years	131	594	1046	1399
18-30 years	290	2028	2319	3404
31-40 years	160	1302	1244	1704
41-50 years	63	512	402	634
51-60 years	19	122	138	187
60+ years	5	21	25	51
Total	668	4576	5174	7379

Table 3 *Age distribution of irregular maritime arrivals between 2008-9 and 2011-12. (HREF6)*

The first pie chart deals with the 2008-09 data.

```
PieChart[{131, 290, 160, 63, 19, 5},
 ChartLabels → {"0-17", "18-30", "31-40", "41-50", "51-60", "60+"}]
```

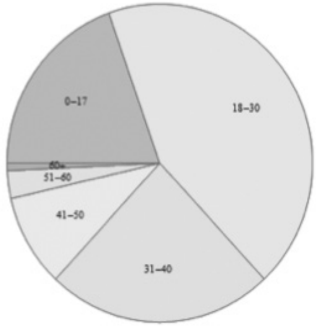

Figure 2. Pie chart of age distribution of irregular maritime arrivals in 2008-9.

Can we combine both in one pie chart?

```
PieChart[{{131, 290, 160, 63, 19, 5}, {1399, 3404, 1704, 634, 187, 51}},
 ChartLabels → {"0-17", "18-30", "31-40", "41-50", "51-60", "60+"}]
```

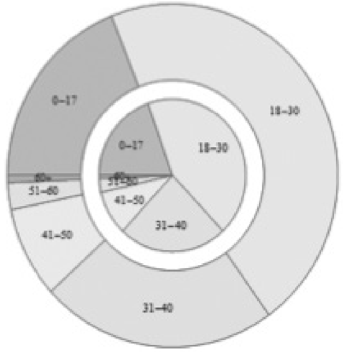

Figure 3. Pie chart of age distribution of irregular maritime arrivals comparing2008-9 and 2011-12.

A paired bar chart can also show this change. However, the pie chart automatically charts ratios whereas the bar chart plots the number in each category.

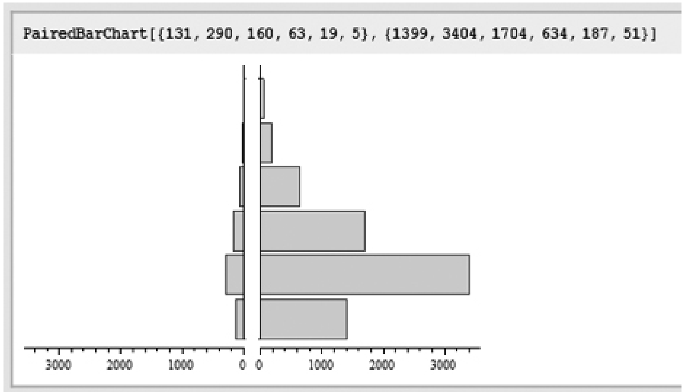

Figure 4. Bar chart of age distribution of irregular maritime arrivals comparing2008-9 and 2011-12. (HREF7).

We can take the string of values for 2008-9 and multiply all parts of the string by 100 and divide by 668 (convert to %).

```
{131, 290, 160, 63, 19, 5} * 100 / 668 // N

{19.6108, 43.4132, 23.9521, 9.43114, 2.84431, 0.748503}
```

This procedure can be applied to both data strings within the BarChart command. We now graph percentages rather than raw totals.

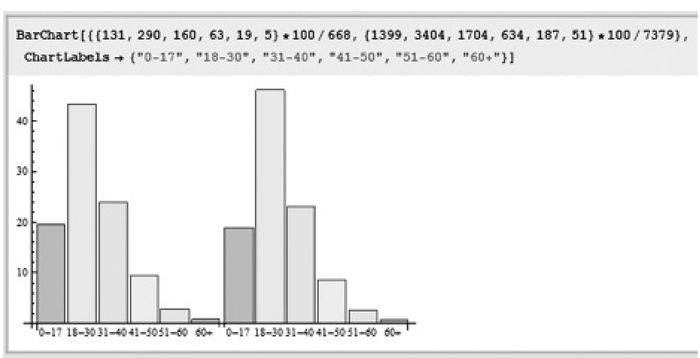

Figure 5. Bar chart of age distribution of irregular maritime arrivals by percentage comparing2008-9 and 2011-12.

Question 5

How far do irregular maritime arrivals have to travel? Many go from Kabul to Christmas Island – how far is that?

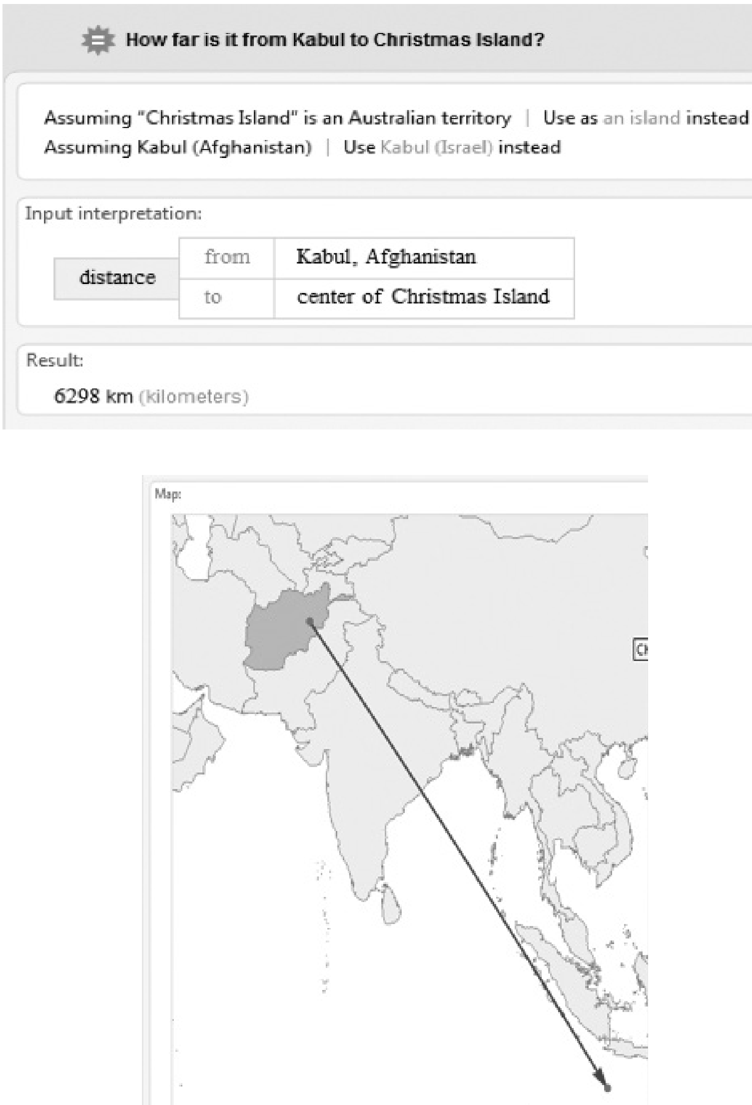

Figure 6. Wolfram Alpha output for question 'How far is it from Kabul to Christmas Island?'.

Plot Linear Relationships on the Cartesian Plane With and Without the Use of Digital Technologies ACMNA193-8

We can answer questions based on historic data and prediction - linear models. Given existing data how could you estimate the number for the following year? How much time do people spend in detention and where are the detention centres located?

Question 1

How many people arrived each year?

The data files used below have been imported directly from spreadsheets.

```
peopleperyear // TableForm
```

1989.	26.
1990.	198.
1991.	214.
1992.	216.
1993.	81.
1994.	953.
1995.	237.
1996.	660.
1997.	339.
1998.	200.
1999.	3721.
2000.	2939.
2001.	5516.
2002.	1.
2003.	53.
2004.	15.
2005.	11.
2006.	60.
2007.	148.
2008.	161.
2009.	2726.
2010.	6555.
2011.	4565.
2012.	17202.

```
ListPlot[Tooltip[peopleperyear],
 PlotRange → {{1985, 2015}, {0, 6000}},
 PlotStyle → PointSize[Medium]]
```

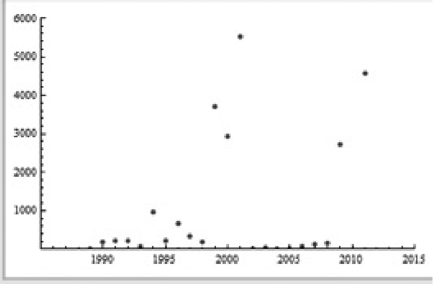

Figure 7. Plot for number of irregular maritime arrivals 1989 to 2012 (HREF7).

```
recentpeople = {{2009, 2726}, {2010, 6555}, {2011, 4565}, {2012, 17202}}

{{2009, 2726}, {2010, 6555}, {2011, 4565}, {2012, 17202}}
```

```
ListPlot[Tooltip[recentpeople], PlotRange → {{2005, 2015}, {0, 20000}},
 PlotStyle → PointSize[Medium]]
```

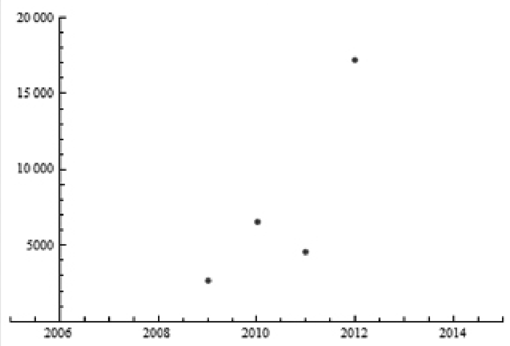

Figure 8. Plot for number of irregular maritime arrivals 2009 to 2012.

In 2013, 13108 irregular maritime arrivals had arrived by the end of June. How many would you predict to arrive in the whole 2013 calendar year?

Method 1

At least 13108!

Method 2

17202 arrived in 2012 so more than 17202 will arrive in 2013

Method 3

13108 arrived in the first half of the year so probably the same number will arrive in the second half. So 26216.

Line of best fit with existing data

```
line = Fit[recentpeople, {1, x}, x]
```

$-8.32335 \times 10^6 + 4143.8\, x$

```
Show[ListPlot[Tooltip[recentpeople], PlotRange → {{2005, 2015}, {0, 20000}},
  PlotStyle → PointSize[Medium]],
 Plot[-8.323347899998237`*^6 + 4143.799999999124` x, {x, 2008, 2014}]]
```

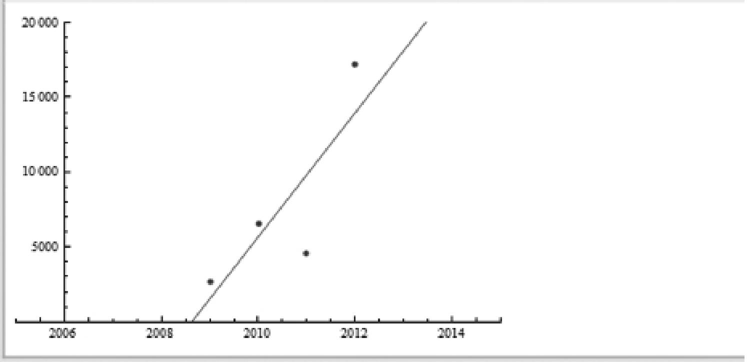

Figure 9. Plot with line of best fit for number of irregular maritime arrivals 2009 to 2012.

Substitute 2013. Adding /. x 2013 replaces x with 2013 in the equation for the straight line.

```
-8.323347899998237`*^6 + 4143.799999999124` x /. x -> 2013

18121.5
```

So, 18121 is the prediction.

Sketch Simple Non-Linear Relations With and Without the Use of Digital Technologies ACMNA296-9

What do we know about the boats - number of boats, size, people per boat, proportion reporting distress?

Question 1

Can we produce a better model of the data if we use a quadratic function?

In this section we are going to find a quadratic model for the data. Since the input is years and we are including x^2 in our model we will be working with numbers like 4000000. To simplify the size of numbers involved we use 1, 2, 3, ... for the set of years. So 1989 becomes 1 and so on.

```
line = Fit[dataarrivals, {1, x, x^2}, x]

2238.79 - 575.737 x + 33.8341 x^2
```

```
Show[ListPlot[dataarrivals, AxesOrigin → {0, 0}, PlotStyle → PointSize[Medium]],
 Plot[33.83 x^2 - 575.73 x + 2238, {x, 0, 25}]]
```

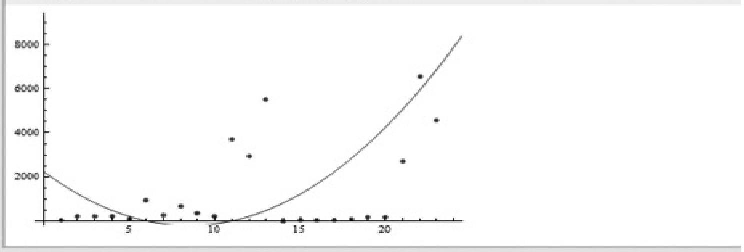

Figure 10. Plot with quadratic model for number of irregular maritime arrivals 1989 to 2012(shown as 1 to 23).

Explore the Connection Between Algebraic and Graphical Representations of Relations Such as Simple Quadratics, Circles and Exponentials Using Digital Technology as Appropriate ACMNA239-10

We use historic data and prediction and consider exponential models.

Question 1

What is the shape of the graph of $f(x) = 2^{a x}$?

We can investigate this by using Manipulate and observing what happens if a varies between -2 and 2 in steps of 0.1.

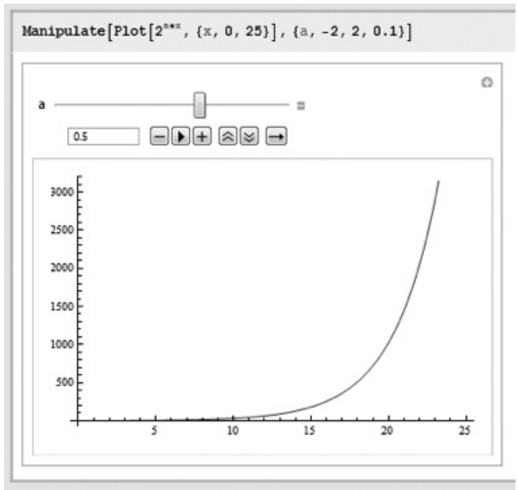

Figure 11. Interactive graph of where a can be varied between -2 and 2 in steps of 0.1.

Question 2

What value of *a* provides a close match to the data?

Simply using trial and error for various values of *a* is one strategy and we see that *a* = 0.57 provides a good match.

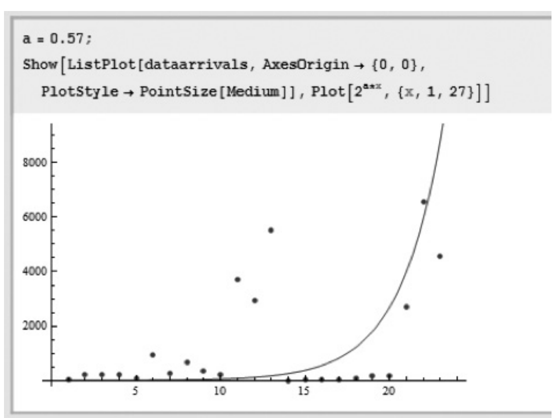

Figure 12. Comparison of graph of graph of with scatterplot of number of irregular maritime arrivals 1989 to 2012 (shown as 1 to 23). Substituting x = 25 provides an estimated number of people in 2013.

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INSIGHTS INTO PROBLEM SOLVING PROCEDURES

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This paper presents an analysis of problem solving procedures of a sample of grade 6 students. The different procedures, both successful and less successful, are analysed and discussed. Examples are used to illustrate and clarify the different procedures used by the students. The procedures are categorized using Polya's problem solving framework and some observations are shared. Implications for learning and teaching are discussed. Some background literature in problem solving is shared to better understand the procedures adopted by the students in problem solving.

Introduction

This study analyses a sample group of 7 students who have gone through 5 years of primary education in Singapore in a neighbourhood school. In the 6th year, the students have to sit for a national examination namely Primary School Leaving Examination (PSLE). This examination is important to the students as their results will determine the secondary schools they will be posted to as well as the stream to which they will be allocated. In Singapore, the students are streamed into Normal Technical (NT), Normal Academic (NA) and Express (Exp) in the secondary schools. Students in the Normal Technical (NT) and Express (Exp) streams take 4 years to complete whereas students in the

Normal Academic stream take 4 - 5 years to complete their secondary education. Thus with such a high stake examination (PSLE) and having gone through 5 years of education, this study aimed to find out how much of Polya's (1973) four stages were demonstrated in the procedures in problem solving to prepare them for the examination. This paper presents an analysis of the extent of application of Polya's problem solving framework in the various categories of the students' work.

Literature Review

Problem solving is at the heart of the Singapore Mathematics syllabus (Ministry of Education, 2007). The approach is to use Polya's 4-step problem solving framework to develop the process in student mathematics learning (Polya, 1973). This comprises helping students to understand the problem, plan the solution, carry out the solution and check the solution.

Understanding the problem requires the students to know what is unknown, what are the data and what is the condition. Students can also draw a figure, introduce suitable notation or separate the various parts of the condition to illustrate their understanding. In devising a plan, students can look for a related problem, recall a theorem that could be useful or look at the unknown and try to think of a familiar problem having the same or a similar unknown. Other strategies include restating the problem, solving part of a problem, keeping only a part of the condition and dropping the other part, or changing the unknown or the data or both. Then the students carry out the plan (Polya, 1973). In carrying out the plan, the students may find some obstacles. Thus the students will have to go back to the earlier stages to analyse their solution. Did they understand the problem comprehensively and accurately? Did they miss out certain steps in their planning? This going back and forward may take a few cycles in problem solving. That is one of the characteristics which we hope to achieve in problem solving, that is, perseverance in their attitude towards problem solving (Ministry of Education, 2007). Finally the students come to the checking stage. Here the students can check the result or the argument. Then the students can attempt to find an alternative solution to the problem. The students should extend their learning beyond the solution to the problem by using the result or method for some other problems. That is the aim for problem solving in our syllabus.

There is also other literature related to problem solving from which we can learn. Pugalee (2004) noted that majority of problem solving behaviours involve execution actions such as carrying out goals and performing calculations. Students who construct global plans are more successful problem solvers and that is stage 2 of Polya's problem solving framework. Students who write descriptions of their thinking are significantly more successful in problem solving tasks than students who verbalize their thinking. Thus, according to this model, students should plan and record their thinking visibly to be more successful problem solvers.

Lawson and Chinnappan (2000) focused on connectedness in helping students to be better problem solvers. Successful problem solvers have a knowledge base that is better organized and more extended. Having this knowledge base helps the students to retrieve the related knowledge faster to connect with the problem. Curiosity, puzzlement, bewilderment, empowerment and disempowerment are affects that have been identified to influence the problem solving abilities of students (Debellis & Goldin, 2006). Curiosity and puzzlement evoke exploratory and problem-defining heuristics and they motivate the solver to better understand the problem. Empowerment acts as an impetus to persevere, take risks, engage with new external and internal representations, ask questions or construct new heuristic plans. However, bewilderment and disempowerment have the opposite affects leading to frustration and inducing negative outcomes associated with maths anxiety and phobia.

Callejo and Vila (2009) studied students' belief systems in problem solving. These fall into two types the first of which is the maths 'you always see' which is mechanical, routine and algorithmic and is used to solve 'the usual problems'. The other type is the maths of 'researching and thinking' in which one solves 'thought problems' that are challenging.

Using the modeling process in problem solving is also in the syllabus (Ministry of Education, 2007). This has commenced at the secondary level. English (2006) has identified students' development in problem solving using the modeling approach namely interpreting and re-interpreting the problem, conceptual growth, mathematisation processes in which they create, use, modify and transform quantities, approaches to quantifying and qualitative data and documenting their own model development and critically reflecting on the models of others.

A technique for assessing mathematical problem solving ability using the Structure of the Learned Outcomes or Responses (SOLO) was used in Collis, Romberg and Jurdak (1986) where five levels were indicated namely prestructual, unistructural, multistructural, relational and extended abstract.

The literature review has provided possibilities of analyzing the problem solving abilities of our students from the definition of a problem (Callejo & Vila, 2009), affects influencing problem solving abilities (Debellis & Goldin, 2006), a sound connectedness in prior knowledge (Lawson & Chinnappan, 2000), the effectiveness of putting the thinking in writing (Pugalee, 2004) and a structure for assessing mathematical problem solving

(Collis, Romberg & Jurdak, 1986). This study will look at problem solving from Polya's problem solving framework.

The Method

The scripts of seven students in a grade 6 class were analysed by looking at 18 questions. These 18 questions were deemed to be problems and the students were allowed to use calculators. Calculators were allowed in the PSLE in 2009. One of the objectives in the implementation of the calculators was to help the students to focus on the problem solving process rather than the computations.

For the four stages in Polya's problem solving framework, some indicators were designed to assist reflection on the procedures used by the students (see Table 1).

Table 1. *Indicators of Polya's four stages*

Stage	Indicators
Understanding	Prior knowledge, familiar question, know what to do, steps indicated, restate the problem/condition, solve part of the problem, break up the problem, missing data, organize data, change the unknown or data, make assumption, work in diagram
Do	Diagram, routine, heuristics, change plan, pause for understanding, give up, apply formula, unitary method, apply notation/symbol, algebra
Check	Ticks and crosses, alternative method, make sense of solution

The understanding stage was to look for evidence of students' attempts at paying attention to the words, phrases and context of the problem. The planning stage was to check if the students did use previous knowledge, express a method or draw/tabulate the data, conditions and unknowns to solve the problem. In the doing stage, the analysis was to determine if the students had a method to solve the problem that was evident. For the checking stage, indicators of check marks, repeated working, another solution or sense making of the solution will be used as evidence of the procedures for checking.

In addition, the teacher also provided some insights into her teaching of the procedures in problem solving in the class. This helps to better understand some of the data in this study.

The problem solving process is analysed based on the overall analysis of all the 18 questions, by topics, by difficulty levels according to the number of marks awarded for the questions and by students' ability to solve the questions correctly, incorrectly or partially correctly.

The Results

Table 2 shows the overall performance of the students showing that they went through the understanding stage for 97.6% of the questions, 50.0% of the questions for the planning stage, 53.2% of the questions for the doing stage and 0.0% for the checking stage. There is consistent use of a procedure mainly shown in the clear highlighting in the questions. The teacher affirmed that in her teaching she did advocate highlighting the questions to help the students to understand the questions (see Figure 1). The percentages for planning and doing are very similar at 50.0% and 53.2% respectively. This is to be expected as students would carry out what they had planned in the solutions.

Table 2. *Overall results*

Understand	Plan	Do	Check
97.6%	50.0%	53.2%	0.0%

In the figure below, the ratio of the area of A to area of B is 1 : 3. The ratio of area C to area D is 4 : 3. If the difference in area between B and C is 25 cm^2, find the area of the whole rectangle.

Figure 1. Highlighting the question

In Table 3 the analysis is based on the topics. The topics tested include whole numbers, measurement, rate, geometry, percentage, non-routine question, fraction and ratio.

Table 3. *Topics*

Topics/ No of questions	Understand	Plan	Do	Check
Measurement (7)	100.0%	53.1%	61.2%	0.0%
Whole numbers (1)	85.7%	71.4%	85.7%	0.0%
Rate (1)	100.0%	0.0%	0.0%	0.0%
Geometry (1)	100.0%	14.3%	26.6%	0.0%
Percentage (1)	100.0%	42.9%	42.9%	0.0%
Non-routine (1)	71.4%	71.4%	42.9%	0.0%
Fraction (4)	100.0%	46.4%	46.4%	0.0%
Ratio (2)	100.0%	71.4%	71.4%	0.0%

Understanding the questions was done at 100.0% except for whole numbers and non-routine questions. At 85.7% and 71.4%, the number of questions for which the students demonstrated their procedures on understanding the question is still high. In the planning stage, whole numbers, non-routine and ratio questions have the most percentage of questions where a plan was devised. Rate showed no planning done by the students for the question. Perhaps the table in the question is a form of planning for the students to carry out the procedures (see Figure 2). In the planning and doing stages, the percentages are similar for rate, percentage, fraction and ratio. For non-routine questions the percentage for planning is 28.5% more than for doing. The non-routine question is not familiar to the students and thus planning a procedure would be expected. Checking is similar to the data in Table 2 where no evidence was found for all the questions. Perhaps it could be a time factor where the students focused on doing instead of checking.

5. The table below shows the charges for water consumption in each household.

First 50 m³ of water	$1.17 per m³
Next 40 m³ of water	$1.28 per m³
Subsequent m³ of water	$1.90 per m³

If Mr Tan uses 124 m³ of water, how much money must he pay?

Figure 2. Rate question with table

The questions were awarded 2 marks, 3 marks, 4 marks or 5 marks. This follows the specification of the PSLE format to prepare the students for the PSLE. A 2 mark question would be deemed to be easier than a 3 mark question and so on. The data in Table 4 show the percentage of the questions for which the students applied the four stages of Polya's problem solving approach

Table 4. *Difficulty levels*

Difficulty level/ No of questions	Understand	Plan	Do	Check
2 marks (5)	100.0%	31.4%	42.9%	0.0%
3 marks (5)	100.0%	42.9%	42.9%	0.0%
4 marks (5)	94.3%	74.3%	68.6%	0.0%
5 marks (5)	95.2%	52.4%	61.9%	0.0%

The understanding stage is still consistently high, ranging from 94.3% to 100.0% as compared to the data in Tables 2 and 3. There is a progression in the percentage of questions where planning is done from the 2 mark questions to the 4 mark questions, increasing from 31.4% to 42.9% to 74.3%. Students were more focused on planning procedures for the more difficult questions. However, there was a decrease to 52.4% for the 5 mark questions. This percentage is still higher than that for the 2 mark and 3 mark questions. Thus it can be seen that students did spend time planning for the procedures for the 4 mark and 5 mark questions. For the doing stage, the data also showed that there was evidence of a more structured approach to doing the 4 mark and 5 mark questions compared to the 2 mark and 3 mark questions. An example can be seen in Figure 3 where the student showed a model drawing and careful tracking of the u (units) and p (parts) to see the relationship between the units and the parts in developing a procedure to solve the question.

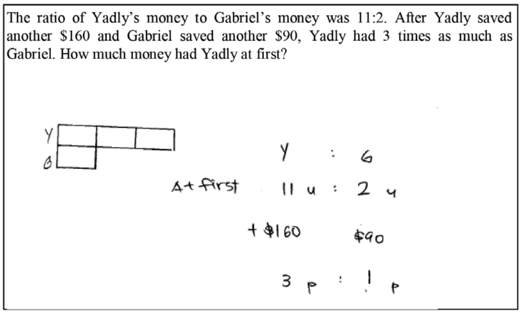

Figure 3. Procedure in the doing stage

In Table 5, we look at the questions that the students did correctly, wrongly or partially correctly. For questions to be deemed to be done correctly, the students must obtain the full marks awarded to the questions. So if a question is awarded 5 marks, then the student must obtain 5 marks to be considered as having done that question correctly. A range of marks from 1 to 4 marks would be deemed to be partially correct and 0 mark to be considered as wrong.

Table 5. *Correct, wrong or partially correct responses*

Responses	Understand	Plan	Do	Check
Correct	100.0%	47.4%	56.6%	0.0%
Wrong	97.6%	53.7%	51.2%	0.0%
Partially correct	77.8%	55.6%	33.3%	0.0%

For the understanding stage, questions done correctly or wrongly had 100.0% and 97.6% respectively whereas only 77.8% of the questions done partially correct showed evidence of attempting to understand the questions. This could be that part (b) of the question was deemed to be easier than part (a) of the question. Thus the answers to part (a) were done by inspection and thus no evidence of any procedure was noted (see Figure 4). For the planning stage, the percentages were 47.4% for correct responses, 53.7% for wrong responses and 55.6% for partially correct responses which differ by 8.2%. Thus almost half the questions had evidence of procedures in planning done. For the doing stage, 56.6% of the questions done correctly and 51.2% of the questions done incorrectly had a structured procedure. Thus we are not able to conclude if the structured procedure helps in solving the questions successfully as both correctly done and wrongly done questions had about 50.0% of the questions with a structured procedure in the doing stage.

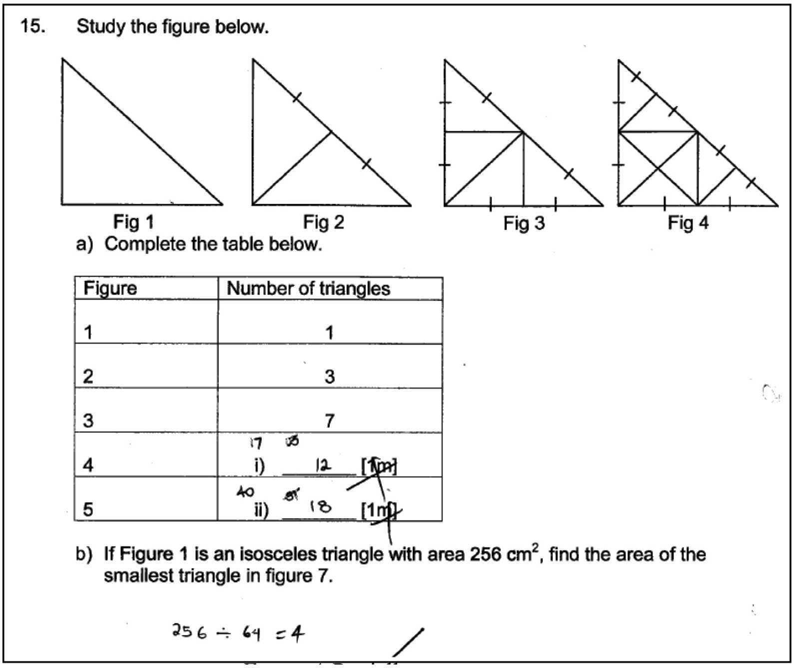
15. Study the figure below.

Fig 1 Fig 2 Fig 3 Fig 4

a) Complete the table below.

Figure	Number of triangles
1	1
2	3
3	7
4	i) 12 [1m]
5	ii) 18 [1m]

b) If Figure 1 is an isosceles triangle with area 256 cm^2, find the area of the smallest triangle in figure 7.

256 ÷ 64 = 4

Figure 4. Partially correct response

Discussion and Conclusion

The study has shown that understanding was demonstrated quite consistently in the students' problem solving approaches but more can be done to educate the students to check their solutions. This is despite the fact that the teacher did remind the students to check their answers. Perhaps the students did the checking mentally as the procedure advocated by the teacher was to work backwards. For the planning and doing stages, the data showed that there is a tendency for both sets of percentages to be close to each other. 68.8% of the differences between planning and doing have a difference of less than 10.0%. From the comment by the teacher, one would expect to see less planning and more doing as the teacher suggested planning mentally as a procedure to solve problems and doing the problems systematically. However from the data, a structured procedure was more evident if there was planning done in the questions. This could be due to the comfort level of the students in seeing their procedures on paper instead of in their minds.

More information on the students' problem solving process can be gleaned by interviewing the students. Alternatively, students can be encouraged to express their problem solving procedures in writing or ticking on the appropriate rubrics with supporting comments.

Polya's problem solving approach can be done in a practical and consistent manner. A procedure of underlining or highlighting the key words can help students to get started to understand the problem. Once the key words are singled out, then a discussion can be started to engage the students in understanding the problem. For planning, drawing a diagram using the model method is a common procedure in planning the solution. The diagram can be populated with the data, unknown and conditions to see the overall problem as well as how best to solve the problem. For the doing stage, brief mathematical statements as a procedure to inform the students of what has been done should help in analyzing the solutions especially when there is a blockage. Checking using the working backwards procedure is one way. Checking to see if the answer is reasonable is another procedure to remind students of sense making in mathematics.

In summary, this study provides some evidence that the students consistently used the procedures to link to Polya's problem solving framework to help them in their examination. However, procedures are necessary but not sufficient in problem solving problem. Understanding the procedures is the key. But who should decide on the procedures? And should there be only one standard procedure?

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A PRIMING INTERVENTION TO IMPROVE GRADE 4 STUDENTS' MATHEMATICAL COMPETENCY AND SELF EFFICACY

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This paper evaluates a priming intervention implemented with eight Grade 4 students identified as low achievers in mathematics in terms of its impact on mathematical competence and mathematical self-efficacy. The intervention was conducted during Term 2, 2013 and was modeled on Sullivan and Gunningham's (2011) Getting Ready intervention that aimed to prepare students for their subsequent mathematics lessons. Results from this study indicate that the priming intervention had a positive impact on students' mathematical achievement and their mathematical self-efficacy.

Background to Study

Low Achieving Students

Low achieving students can be described as children who underperform in the classroom, are slower to make progression in their learning, and have substantial difficulty meeting educational expectations and standards (Van Auker-Ergle, 2003). Research suggests that as many as one in every three children in Australian classrooms may be failing to meet minimum standards in mathematics (Ferrari, 2012; Stanley, 2008). The problems these particular students may have include problems with computation skills, word problems, mathematics vocabulary, and reasoning skills (Milton, 2001).

Low achievers who have had little opportunity to develop positive views of their mathematical ability often experience low selfefficacy. Children with low mathematical self-efficacy display negative responses in all three domains of Tait-McCutcheon's (2008) model of self-efficacy; affectively, cognitively and conatively. They avoid challenging learning tasks, engage in negative thought patterns, and convince themselves they cannot succeed (Gray & MacBlain, 2012). Subsequently, low achieving students have trouble staying focused and engaged, compounding their difficulties with mathematics. Gervasoni (2003) argued that this loss in confidence is the main contributor to the knowledge gap between students, and that the normal learning experiences provided within the class do not allow low achievers maximum participation.

The construct of mathematical self-efficacy applies specifically to the context of mathematical learning or mathematical tasks. Research has found that self efficacy is an important factor in mathematical performance (Pajares & Graham, 1999; Stanley, 2008). Mathematical self-efficacy, or the belief a student has about their ability to solve mathematics problems, has been found to be the strongest predictor of mathematical achievement, stronger than both intrinsic motivation (Stanley, 2008) and general cognitive ability (Pajares & Graham, 1999). Students' mathematical self-efficacy begins to form in the early years of mathematical learning, making the beginning years of schooling extremely important to building positive attitudes towards mathematics (Tripathi, 2009).

Priming Interventions

Sullivan and Gunningham (2011) reported on the impact of an intervention for low achieving students that focuses on preparing students for the mathematics lesson they will encounter directly following the support session. We refer to this form of "pre-emptive" support as Priming. Priming sessions focus on the identified needs of low achieving students by providing explicit instruction, individual attention, repetition and reinforcement, and experiential learning techniques. In each priming session, tutors highlight and familiarise students with key mathematics vocabulary, discuss and align prior knowledge, and explicitly model the activities of the next lesson. Sullivan and Gunningham (2011) noted that the benefits of this type of intervention included improved confidence and a change in the way students thought about their learning.

Based on Sullivan and Gunningham's (2011) conclusions, we anticipated that a priming intervention could improve students' mathematical self-efficacy across each of the domains: affective, cognitive, and conative. We believed that priming would work to address students' affective self-efficacy by providing students with explicit modeling and supporting individual practice of mathematical concepts. In this way, students are afforded opportunities to experience success in a safe, encouraging environment, thus working to build confidence. Furthermore, priming potentially targets cognitive self-efficacy by introducing students to the format, concepts, and processes that will be the focus of the subsequent lesson. We envisioned that this familiarity would allow students a greater awareness of their thinking and would increase engagement. Finally, we thought that priming would impact on increasing the conative or motivational aspects of a student's self-efficacy. Provision of tutor support encourages students to persist and expend effort to reach successful outcomes. Through experiencing success within the intervention session, it is expected that students will display higher levels of motivational behaviour within the regular mathematics lesson, including persistence and effort.

In this paper, we report on our study that explored the impact of a priming intervention on students' affective, cognitive, and conative self-efficacy. A group of Grade 4 students were chosen as participants for this study because this age has been emphasised as a critical stage in the development of students' self-efficacy (Metallidou & Vlachou, 2007). An intervention with students at this age level has the potential to improve students' self-efficacy, specifically their belief in their own ability to successfully perform mathematics learning and improve motivational behaviours.

Method

The study utilised a mixed methods research design, as described by Crewswell (2012). The intervention was conducted throughout Term 2, 2013, in a Central Victorian government school. Eight Grade 4 students (6 boys and 2 girls) were selected by their teacher to participate in the program. The program was structured as two weekly 15 minute sessions that were implemented prior to the scheduled mathematics lesson for the day. The students were placed into two groups of four, and a researcher-tutor was allocated to each group. The focus for the term was on number and worded problems. The Priming sessions focused on addition, subtraction, multiplication, and solving related worded problems.

Quantitative data were collected to measure the effects (if any) of the intervention on student competence and self efficacy. These data consisted of student responses to a pre- and post- survey of beliefs about mathematical competence that was comprised of 15 likert-scale items that related to the affective, cognitive, and conative domains of self-efficacy, results generated by on-demand testing, and a pre- and post- mathematics quiz. Qualitative data were collected to examine in more detail the student responses to the intervention. Qualitative data consisted of students' responses to 5 open-ended questions on the pre- and post- survey. We now present and discuss our findings.

Findings and Discussion

Mathematical Achievement

Students' mathematical progress as measured by the On-Demand testing results from the start of Year 4 to midway through Year 4 was compared to their progress from midway through Year 3 to the start of Year 4. Although the limited sample size precludes making generalisations about these results, there is a trend that suggests that the intervention had a positive impact on students' achievement. For the "Midway through Year 3" and "Start of Year 4" results, all students were below the expected level according to VELS progression points. Results for Midway through Year 4 indicate that 3 of the 8 students are achieving at the expected level, with others demonstrating consistent growth. Table 1 shows the mean progression point scores for this group of students as measured by On-Demand testing. Note that the average progression for the group between Midway through Year 3 and Start of Year 4 was 0.05. However, the mean progression from the start to midway through Year 4 increased to 0.6. As noted above, it is important to treat these statistics with caution due to the size of the sample. However, the results do indicate a promising trend.

Table 1. *Mean Progression Point Scores (n=8)*

	VELS Level	Mean	Stand Dev
Mid Yr 3	2.25	1.75	0.3
Start Yr 4	2.5	1.8	0.3
Mid Yr 4	2.25	2.4	0.5

The students completed a pre- and post- quiz. The pre-quiz comprised a series of mathematical questions developed with reference to the content and achievement standards for Level 4 of the Australian Curriculum which is the expected benchmark for Grade 4 students (Australian Curriculum Assessment and Reporting Authority (ACARA), 2012). To this end, questions relating to the domains of Number and Algebra, Measurement and Geometry, and Statistics and Probability, were included in the quiz. The intervention was designed to prime students for their upcoming mathematics lessons, and during Term 2, the focus of the Year 4 mathematics curriculum was on addition, subtraction, and solving worded problems. The post- quiz was redesigned to reflect this focus. Table 2 shows the students' mean scores on the three addition and subtraction items on the pre-quiz, and their mean scores for the 20 items on the post-quiz. We acknowledge that it is inappropriate to generalise any conclusions from these statistics due to the small sample size and the fact that the pre-quiz contained substantially fewer addition and subtraction problems. However, what is clear is that 5 of the 8 students were unable to answer any of the pre-quiz addition and subtraction questions correctly and this may indicate that it is unlikely that their results would have improved with the inclusion of more questions of this type.

Table 2. *Mean Pre and Post Quiz Scores (n=8)*

	Mean	Stand Dev
Pre-Quiz	0.17	0.25
Post-Quiz	0.66	0.3

Mathematics Self-Efficacy

Mathematical self-efficacy was defined as having an affective, conative, and cognitive domain, and this guided the development of survey and interview questions and was the basis for the data analysis. Table 3 presents data from the pre- and post- self-efficacy survey that presented statements for students to indicate their agreement or disagreement. For positively worded statements, student responses were coded with 1 for disagree, 2 for a neutral response, and 3 for agree. For negatively worded statements (noted with an asterix in Table 3), the coding was reversed. Therefore, mean scores closer to 3 were interpreted as indicators of higher self-efficacy and a shift in mean scores to a higher score was interpreted as an improvement in self-efficacy. The mean scores for the majority of items on the pre- and post- surveys indicated a positive change in students' mathematical self-efficacy.

Table 3. *Students' Mean Scores for Mathematics Self Efficacy (n=8)*

	Survey 1	Survey 2
I am confident when I do maths	2.4	2.9
Maths is hard for me *	1.9	2.4
I believe I can do well in maths	2.9	3

	Survey 1	Survey 2
When I am given a maths problem, I am confident that I will know how to go about answering it	2.1	2.4
In my class, I see myself as good at maths	2.3	2.5
When a maths problem is difficult for me to solve, I just put in more effort	2.8	2.8
I will work as long as necessary to solve a challenging maths problem	2.8	2.4
I work and try hard in maths even if I don't like what we are doing	2.6	2.8
When maths is hard, I either give up or only do the easy parts*	2.4	2.5
I try as hard in maths as I do in my other subjects	2.6	2.7
I have a hard time concentrating on maths and am easily distracted *	1.8	2.3
I often feel so bored in maths that I quit before I finish my work *	2.8	2.4
I am interested in what we learn in maths	2.6	2.7
In maths I try to relate what we are learning to what I already know	2.6	2.9
I try to develop my own ideas from what we are learning in maths	2.4	2.8

Table 4 presents the mean scores for the student responses to the survey questions grouped by mathematical self-efficacy domain. The students demonstrated a positive shift in their affective (engagement and self-belief) and cognitive (engagement) mathematical self-efficacy. The mean scores that represent their pre- and post- conative self-efficacy (motivation and persistence) indicate that this remained constant over the course of the intervention.

Table 4. *Mean Scores for Mathematical Self-Efficacy by Domain*

	Survey 1	Survey 2
Affective	2.3	2.6
Conative	2.6	2.6
Cognitive	2.4	2.6

As noted, due to the small sample of participants, these differences in mean scores were not significant. However, they were indicative of the trend evident in the data generated from the open ended questions which demonstrated that students' mathematical self-efficacy was enhanced as a result of participating in this program. The following section discusses the qualitative data in relation to the three domains of mathematical self-efficacy.

Affective Self-Efficacy

The students responded to an open ended item on the pre-survey that asked *How do you feel when you are doing mathematics?* Students articulated a range of emotions in response, ranging from "a little bit scared" (Student 4), "a bit nervous… and sometimes bored and distracted" (Student 7), and frustrated, to "pretty happy" (Student 3), "sometimes confident" (Student 4), and "really confident" (Student 6). Students tended to qualify their responses and indicated that their feelings depended on the topic or whether the content was new. Student 8 responded: "Depends on what we are doing in maths…If we are doing times tables, then I feel a bit weird". Interestingly, a number of students described a physical sensation when asked to describe how they felt about mathematics. Student 4 noted that she sometimes experiences a "weird feeling in the belly because most of it (mathematics) I don't know" and Student 1 noted that sometimes she "can't concentrate and my head spins around".

Student responses to the open ended items on the pre-survey also indicated they did not think they learned mathematics easily, although this also depended on the content. Student 7 reasoned that he did not learn mathematics easily because he needed concepts "to be explained a couple of times". It appears that Student 7's awareness that he needs to hear explanations a number of times, has impacted on his self-belief about his competence. A number of the students noted that it took them "a while" to understand as a way to justify why they thought they did not learn mathematics easily. This was evidenced in Student 8's response: "sometimes it takes a little bit but then I understand". We interpreted this as evidence of students' self-belief about their competence (part of the affective domain), but also evidence of their metacognition – their awareness of how they think and learn – which relates to their cognitive self-efficacy.

Students' lack of confidence in their mathematical ability was also indicated in their answers to the question *What do you do when you don't know how to do something in maths?* Both of the girls indicated that they will ask for assistance from the teacher, but only when other students are not around. Student 1 commented that she does not ask for help when she is in a group, and Student 4 responded that she will "wait until everyone goes so I can ask for help without everyone around". This indicated that, for the girls at least, their lack of confidence and negative self belief impacted on their willingness to actively seek out support.

Students' responses to the question *How do you feel when you are doing mathematics*? on the post-survey were quite different to their responses to this question prior to the term long intervention. Three of the students noted that they now feel confident with Student 1 commenting: "I feel great because I am confident". Student 6 noted that he feels confident, and reasoned: "I have more things in my head and more ideas to get the answers. Doing extra maths really helped me". Other responses included feeling happy, feeling good, and feeling excited. One student did note that she feels "OK, because now I know a bit more and I have improved", however, she did add that she still felt "a bit scared" of mathematics.

It was quite clear that students' confidence had improved as a result of the intervention as they were able to explicitly articulate this themselves. There was also evidence that their self-belief in the competence had improved. Student 1 commented: "I definitely think it has gotten easier because I know more than I think". These responses demonstrate that the priming intervention impacted positively on students' affective self-efficacy and the responses support the quantitative data that indicated an increase in mean response to questions relating to affective self-efficacy on the pre- and post-survey.

Conative Self-Efficacy

Students' conative self-efficacy was conceptualised as their willingness to persist and their level of motivation. When asked, on the pre-survey, What do you do when you don't know how to do something in maths?, the most common responses were that students would often skip questions if they were unsure or ask for assistance. Both responses were also the most common on the post-survey, and there was limited evidence that students' conative self-efficacy had changed. Indeed, this supports the quantitative data that showed no change in mean score for the students' conative self-efficacy on the pre- and post- survey.

Cognitive Self-Efficacy

Cognitive self-efficacy was interpreted as student engagement and their metacognition. Most students suggested that they need to listen better, concentrate and not get distracted in response to a question that asked: *What do you think would help you to be better at maths*? Student 7 commented: "If we are sitting learning maths, then I can't sit still, I get bored and get distracted". Student 4 noted that the complexity of the mathematics impacts on her ability to engage; she noted: "if it gets too hard, I switch off". Student 1 noted that she has "sensitive concentration", and it was a little unclear what she meant by this. Perhaps, like Student 7, she is easily distracted. Or perhaps, like Student 4, she finds it difficult concentrating if the content is difficult. Student 1 also noted that the pace of mathematics lessons is often too quick for her and that she needs extra time and "lots of attention with explaining things". What is particularly interesting about these comments is the level of student awareness of what they need to do to learn mathematics and what it is that is constraining them. This is evidence of their metacognition – their thinking about thinking – and we classified these responses as evidence of their cognitive self-efficacy.

Students' responses to the post-survey questions demonstrated an increase in their cognitive self-efficacy and this was also demonstrated by an improvement in the mean scores for this domain. A number of students noted that the intervention had helped them because it had provided extra help and extra time. Student 7 commented: "I feel excited when doing maths because I can take as long as I like". The intervention was also attributed as assisting with students' understanding and by helping students "learn more strategies", and this was further evidence of the students' metacognitive development. Student 5 noted that to be better at mathematics you need to have the confidence to do it. Once again, the level of insight that the students provided about their mathematics learning was quite surprising.

Conclusion

This study found evidence that a priming intervention impacts positively on students' mathematical achievement. The development of students' mathematical competence was evident in their improvement in both their results from On-Demand testing and their results on a pre-and post quiz. This study also found that students' mathematical self-efficacy increased as a result of the intervention, particularly within the domains of affective and cognitive self-efficacy. Loss of confidence, an element of affective self-efficacy, is a major factor that contributes to the knowledge gap between students (Gervosoni, 2003). Student responses to the pre-survey indicated that their confidence and self belief in their mathematical ability was adversely impacted by their experiences learning mathematics. The intervention provided extra time and practice for these students, and this may have been one factor that contributed to the improvement of participants' confidence and self belief. Throughout the study, students also demonstrated an awareness of how they best learned mathematics and what constrained their learning. Their capacity to think metacognitively about their mathematics learning appeared to develop further as a result of the intervention as well. Engagement and metacognition are considered elements of cognitive self-efficacy. However, there was limited evidence that the intervention impacted on students' conative self-efficacy (conceptualised as student motivation and persistence). Overall, these were promising results.

The priming intervention, modeled on Sullivan and Gunningham (2011), was designed so the students were only withdrawn from classes for 30 minutes a week, was focused on content that was to be the focus of their next lesson, and specifically catered for low achieving students in terms of pace and repetition. It is envisaged that a program such as this could be embedded into a normal mathematics lesson.

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INTEGRATING LEARNING THEORIES INTO THE DESIGN, IMPLEMENTATION AND EVALUATION OF RICH MATHEMATICAL TASKS

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Third year pre-service teachers integrated learning theories and practical knowledge when designing, implementing and evaluating rich mathematical tasks. Pre-service teachers completed a combined project within a theory based subject – Theories of Learning, and a mathematical pedagogical and content based subject – Teaching Mathematics, whereby they designed a rich mathematical task designed to foster deep learning and adapted this in light of theories relating to student autonomy, self regulation, and metacognition. The pre-service teachers implemented their task with groups of primary students, and evaluated the effectiveness of the task in terms of students' responses. This paper presents the mathematics tasks that were developed, and discusses the preservice teachers' successes and challenges.

Deep Learning in Mathematics

The Australian Curriculum – Mathematics (Australian Curriculum Assessment and Reporting Authority (ACARA), 2013) is underpinned by Kilpatrick, Swafford, and Findell's, (2001) conception of the development of mathematical proficiency; procedural fluency, conceptual understanding, strategic competence, and adaptive reasoning. The rationale for school mathematics is for students to access "the power of mathematical reasoning and learn to apply their mathematical understanding creatively and efficiently" (ACARA, 2013). It is clear that a major focus of The Australian Curriculum – Mathematics, is on the application of higher order thinking and problem solving, and the development of conceptual understandings of mathematics. We refer to this as deep learning in mathematics.

One of the fundamental enablers of deep learning in mathematics is the mathematical task (Henningsen & Stein, 1997; Sullivan, 2011). Task choice or development becomes crucial in a mathematics lesson designed to foster deep learning. Planning and implementing mathematical tasks are considered to be key and complex elements of teachers' work (Ball & Bass, 2000; Sullivan & Mousley, 2001) as they strive to teach for understanding. Tasks that encourage deep learning may include open ended problems, applying mathematical knowledge and processes to a task, using multiple ways to represent and communicate thinking and learning processes, and developing multiple solutions. These approaches can be enhanced by integrating a number of learning approaches consistent with deep learning – such as the pedagogical concepts of student autonomy, meta-cognition and self regulation.

Beswick, Swabey, and Andrew (2008) note that student autonomy is a key feature of supportive mathematics classrooms. Classroom contexts that support autonomy do so most effectively through the provision of choice and the removal of external controls (Deci & Ryan, 1994; Stefanou, Perencevich, Di Cintio, & Turner, 2004). Stefanou et al (2004) explain that autonomy-supportive practices, in particular those that allow students to use their "unique ways of solving problems to make meaningful conclusions" (p. 101), can engage students in learning for leaning's sake. They argue that when students' main drive is to understand, this encourages the "most self-determined form of extrinsic motivation" – integrated regulation. Related to autonomy is self-regulation referring to "self-generated thoughts, feelings, and behaviours that are oriented to attaining goals" (Zimmerman, 2002, p. 65), and the effectiveness depends on students' perceived efficacy and intrinsic interest. Zimmerman (1998) noted that self-regulated learners "believe that academic learning is a proactive activity, requiring self-initiated motivational and behavioural processes as well as metacognitive ones" (p. 1). In terms of task design, self-regulation and autonomous approaches would include students planning, enacting, monitoring and adapting their own cognitive strategies in order to effectively complete the task.

Self-regulated learning behaviours also involve metacognitive processes. These are critical for the development of deep mathematical understanding through students autonomously analysing mathematical problems, examining personal mathematical thinking, and explaining and justifying mathematical reasoning (Pape et al., 2003). In terms of task design, metacognitive processes are evident in strategies for building student knowledge of one's self as a mathematical learner, knowledge of the mathematical task and appropriate strategies to complete the task; awareness of the mathematical experience and development of a learning language grounded in mathematics is also an important component.

These learning concepts and models have been identified as contributing to deep learning across domains, but here we demonstrate and evaluate how we referred to these explicitly with the intent of enhancing the design of the mathematical tasks to encourage students to engage deeply and learn conceptually. We present examples of our tasks, and discuss what we learned about task design from student responses. The following section presents examples of our redesigned mathematical tasks and we share how we interpreted and integrated the pedagogical approaches of metacognition, student autonomy, and self-regulation.

Developing Tasks for Deep Learning

In this section we share the mathematical tasks we designed, adapted, and then implemented with a group of students, as we developed our own understanding of different learning theories and approaches. We acknowledge that we have drawn our inspiration for our tasks from a variety of sources. Some of the tasks were adapted from problem solving activities that we explored during tutorials, and others have been inspired by our observations in classrooms whilst on practicum.

Heidi's Task

There are horses and humans in a paddock and I count 86 legs altogether. How many horses and humans could there be in the paddock?

My task was an open ended question that had multiple solutions and solution pathways. I encouraged students to use multiple strategies to solve the problem as well as using different representations of the problem – concrete, pictorial, and arithmetical. I designed the task to focus on communicating mathematical thinking and learning processes by focusing students' attention on the strategies they used rather than correct solutions. Students were encouraged to try different methods to solve the problem and to find a range of solutions. As I designed the task and planned for its implementation I focused on the theories of metacognition, and self-regulated learning. I thought it was important that the task should allow students to be able to think metacognitively to foster deep learning and so I modeled how I would solve the problem whilst explaining my thinking as I solved the problem. I also provided opportunities for students to explain their solutions and their thinking, and then asked students to solve the problem again. This provided them with an opportunity to refine their strategies or try different ones, and therefore promoted self-regulation as they evaluated and adapted their learning strategies.

A particular challenge during implementation of the task was that students were unable to differentiate between explaining their solutions and explaining their thinking. When asked to explain their thinking behind their solutions, they often described what they did instead of describing their thinking and learning. I think that it is also important for teachers to model solution strategies and model thinking processes to facilitate self regulation and metacognition.

Stephen's Task

My task was designed for Grade 5/6 students and focused on perimeter and area. Students were asked to explore different designs for a garden bed, firstly with a fixed perimeter, and then of any size. The task purposefully incorporated aspects of three learning theories – Student Autonomy, Metacognition and Self-regulated Learning. Students then physically mapped out their shape in the school's garden. The task involved students working autonomously, relating conceptual knowledge to real life through representation and application, and demonstrating awareness of their thinking processes. Figure 1 presents my task along with annotations to indicate how the related learning theories impacted on the task design.

Stephen's Task

Learning Outcome: Students will be able to calculate perimeter and area of various shapes.

Assessment criteria: Student ability to correctly calculate perimeter and area and recording and self-review of goals.

"Task analysis involves goal setting and strategic planning." (Zimmerman, 2002, p. 68) Encouraging students to set and review goals and evaluate their progress encourages self-regulation.

"...self-recording personal events" (Zimmerman, 2002, p. 68). Asking students to record not only their results but why they made that design is developing their skill of self-regulation

Classroom contexts that support autonomy do so most effectively through the provision of choice (Deci & Ryan, 1994; Stefanou, et al., 2004).This task allows students greater choice while the need to calculate the area and perimeter, which relates back to the desired learning outcome, will help to build their expertise of these

The school wants you to create a new vegetable garden and they have provided 10m of flexible edging that can be cut to any length.

What shape would you make? Set a goal for the number of designs you will test.

What is the area of the shape you have made? Remember the maximum perimeter is 10m.

So that as many plants as possible can be grown in the garden, try to find the largest area you can make with a perimeter of 10m? You will need to try some different designs.

Keep a record of your designs, their perimeter AND area and write down how you chose that design. I want to know what you were thinking.

The garden will be located in the area shown above. If you had no limit to the amount of edging you had, what shape and size would you make? How much edging do you need for your design and what is the area?

Reread your goals and ask yourself if you have met your goals and answered the questions.

In your book, explain your choice of shape and how the shape affected the area. Explain what you were thinking while doing this. If you have trouble writing this down, explain it to me in words.

"A deep approach to learning involves an intention to understand and impose meaning" (Smith & Colby, 2007, p. 206). The intention is for students to understand that changing the shape will change the area even if the perimeter remains the same.

"In the deep approach the intention to extract meaning produces active learning processes that involve relating ideas and looking for patterns and principles ..." (Entwistle, 2000, p. 3). This task will help relate the idea of connections between perimeter and area and encourage students to think about the principles involved.

"...comparison of self-observed performance against some standard" (Zimmerman, 2002, p. 68) Reviewing goals is a form of self-evaluation, an attribute displayed by self-regulated learners.

Tarricone (2011) defines metacognition as "knowledge about one's own cognition rather than the cognitions themselves" (2011, p. 1) and this task requires students to consider their thinking. Here students are encouraged to record or verbalise their thought processes as "Reflection including verbalisation is essential for the development of metacognitive strategies" (Tarricone, 2011, p. 27).

Being able to explain the procedure used is an "... expression of one's own knowledge in relation to strategies used or performing a cognitive task" (Whitebread et al., 2009, p. 79) and is an important aspect of metacognition. Encouraging students to explain their strategies helps them to understand their thinking and provides the teacher an opportunity to assess their understanding.

Figure 1. Stephen's task.

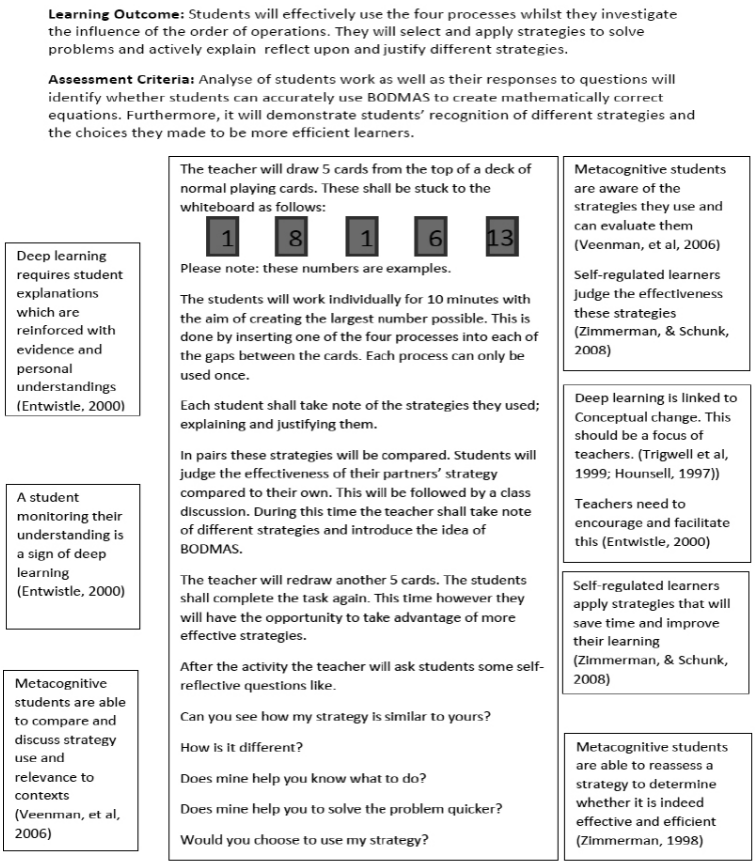
Although it took longer to plan and prepare than a simpler surface task, (or low level task), the benefits of this planning became evident during the implementation of the task. A significant portion of the planning time was spent becoming familiar with the relevant theories, which would not be necessary for planning future tasks, however it would be beneficial to devote time to keeping up to date with current theories.

I believe having choice is integral to student autonomy and impacts positively on their learning. The students responded very well to being able to choose the design of their garden which meant they could choose the complexity of the shape whose perimeter and area they had to calculate and so were able to create a design they could be successful with. I think having cognitive autonomy, and being able to devise their own strategies and understanding, enhanced their engagement with the task and subsequently their learning. The fact that they all produced different designs demonstrated their autonomy. As also discovered by my fellow pre-service teachers, students initially struggled with meta-cognitive practices and found it difficult to describe their thinking, instead describing what they had done. More probing questions and demonstrating the process of verbalising thoughts helped to clarify this and enabled students to respond more accurately about their thinking. This practice is one that would improve with practice and appropriate modeling, providing students with an appropriate vocabulary.

Scott's Task

My task required the teacher to draw five cards at random from a deck of playing cards, which were then displayed at the front of the room. Students then had to use addition, subtraction, multiplication and division processes to create the largest number possible. The intention of the task was for students to investigate the influence of the order of operations on equations. Students would explain and justify their strategies as part of the class discussion. The activity was then repeated with different cards, giving students the opportunity to revise their approach.

The task was designed to stimulate student interest, develop understanding of the order of operations, and develop an ability to solve new and familiar problems. A key part of the task was guiding students, through metacognitive awareness and reflective discussion, to recognise and apply efficient solution pathways. Figure 2 presents my task along with annotations to indicate how the related learning theories impacted on the task design.



Learning Outcome: Students will effectively use the four processes whilst they investigate the influence of the order of operations. They will select and apply strategies to solve problems and actively explain reflect upon and justify different strategies.

Assessment Criteria: Analyse of students work as well as their responses to questions will identify whether students can accurately use BODMAS to create mathematically correct equations. Furthermore, it will demonstrate students' recognition of different strategies and the choices they made to be more efficient learners.

Deep learning requires student explanations which are reinforced with evidence and personal understandings (Entwistle, 2000)

A student monitoring their understanding is a sign of deep learning (Entwistle, 2000)

Metacognitive students are able to compare and discuss strategy use and relevance to contexts (Veenman, et al, 2006)

The teacher will draw 5 cards from the top of a deck of normal playing cards. These shall be stuck to the whiteboard as follows:

1 8 1 6 13

Please note: these numbers are examples.

The students will work individually for 10 minutes with the aim of creating the largest number possible. This is done by inserting one of the four processes into each of the gaps between the cards. Each process can only be used once.

Each student shall take note of the strategies they used; explaining and justifying them.

In pairs these strategies will be compared. Students will judge the effectiveness of their partners' strategy compared to their own. This will be followed by a class discussion. During this time the teacher shall take note of different strategies and introduce the idea of BODMAS.

The teacher will redraw another 5 cards. The students shall complete the task again. This time however they will have the opportunity to take advantage of more effective strategies.

After the activity the teacher will ask students some self-reflective questions like.

Can you see how my strategy is similar to yours?

How is it different?

Does mine help you know what to do?

Does mine help you to solve the problem quicker?

Would you choose to use my strategy?

Metacognitive students are aware of the strategies they use and can evaluate them (Veenman, et al, 2006)

Self-regulated learners judge the effectiveness these strategies (Zimmerman, & Schunk, 2008)

Deep learning is linked to Conceptual change. This should be a focus of teachers. (Trigwell et al, 1999; Hounsell, 1997))

Teachers need to encourage and facilitate this (Entwistle, 2000)

Self-regulated learners apply strategies that will save time and improve their learning (Zimmerman, & Schunk, 2008)

Metacognitive students are able to reassess a strategy to determine whether it is indeed effective and efficient (Zimmerman, 1998)

Figure 2. Scott's task.

This high-level mathematical task allowed for multiple solution pathways and aimed to foster students' high-level cognition through the incorporation of elements that encouraged metacognition and self-regulation. Elements of the task, namely off-line think-aloud statements, written explanations of strategies and evaluative questioning, honed the metacognitive ability of students. Self-regulated learning was facilitated through the provision of cognitive autonomy for the students in terms of provision of choice in strategy use, and adequate time for reflection. Students were able to reattempt the task; as such they were able to evaluate, revise and implement their strategies.

As my peers also discovered, students lacked the language to explain their thinking processes and thus provide an insight into their metacognitive ability. Students also experienced difficulty analysing a single strategy to identify its effective components. However, when presented with multiple strategies, students could discuss and compare strategies.

As a preservice teacher, I found some challenge in formulating and facilitating the task. In order to teach with cognition in mind I had to have a sound understanding of each of the learning theories I was incorporating into the tasks. I needed to know what enabled each of the learning approaches and what signals demonstrated their success. Moreover I had to be aware of how each of the theories interacted with each other. With this in mind a great deal of time was spent on the planning stage and devising approaches to make students cognition visible.

Bethany's Task

With a partner, write a number story that begins with the number 84 and ends with the number 2. You must use each of the four operations at least once.

My task was an open-ended task that focused on the four operations and allowed for multiple solutions and pathways. The task required students to use multiple strategies to create a number story starting at 84 and finishing with 2 which used each operation at least once. As students completed the task, they were expected to discuss and justify their thinking and their choices as a way to promote mathematical reasoning, metacognition, and self regulation. As students discussed and made strategic choices they also annotated their solution to record their thinking about thinking and this provided an opportunity to develop their metacognitive skills. The lesson structure was designed to provide opportunities to share, reflect on and evaluate different strategies and this was guided by my understanding of the key features of self-regulation.

Students struggled to articulate their thinking as they completed the task and this highlights that it would be beneficial to provide opportunities for students to practice this aspect of metacognition as a way to encourage deep learning in mathematics. Listening to and watching the groups of students working together provided evidence of students' reasoning. Interestingly, the students were unaware that their discussions demonstrated their reasoning and self-monitoring and were a way to provide a justification for each step of their solution path. Prompting by the teacher assisted the students to annotate their solutions by recording their verbal explanations and some students began to understand that being able to discuss and justify a solution path enabled their learning. This is where the connection between metacognition and self-regulation became clear for me; as students were thinking about their thinking they recognised what assisted their learning (in this case, discussion with others), which provided them with a strategy that they could draw on when solving another problem. By seeing that talking though a solution provides a deeper level of understanding, students were demonstrating self-regulation when choosing to do this to assist their understand and solve a problem.

Chelsea's Task

How many basketball widths would make a kilometre?

The theoretical concepts of student autonomy and metacognition were drawn on in the design and implementation of an open-ended problem with a class of grade five and six students. The problem – "How many basketball courts laying side by side would be required to make 1km?" – required the students to predict the width of one basketball court in metres using their own informal measurement strategies.

As I designed the task, I considered principles related to deep learning, student autonomy and metacognition, and this influenced aspects such as an emphasis on processes rather than products in the language I used. I asked students to "implement a strategy" and "explain the strategy you have chosen and the steps you took in order to find the solution" rather than simply "solve the problem and explain how you solved it" to demonstrate that I valued their strategies. Through incorporating visual diagrams, encouraging the use of tangible resources and breaking down the question to establish what the mathematical equation could look like, the students were able to grasp the idea and employ an effective strategy.

I found that allowing the students to explain their strategies provided each student a sense of ownership over their learning and gave each strategy value within an environment that valued their contributions. Students who were able to complete the task independently acknowledged that the learning process was far more important than finding the end result. They were also able to link this new experience to knowledge that they already held in order to extend their thinking and make worldly connections.

Meagan's Task

A group of students are on tour at a wildlife park that is home to dingoes and emus. During the day they see 25 animals with a total of 80 legs. How many emus and dingoes did they see?

The design and implementation of the task was informed by theories relating to self regulation and metacognition. Students worked individually to trial strategies and then discussed their solutions and strategies in pairs. On completion the group discussed and demonstrated their strategies. Asking students to share and explain their reasoning and their solution methods provided opportunities for self-regulation. This was the basis for asking students to pose similar problems and try different strategies to solve as it prompted them to evaluate the effectiveness of each strategy – a key aspect of self-regulation.

I found that the students were able to talk to each other in pairs about how they solved the problem and could justify their choice of strategy, but when I asked them to explain their solutions they found it difficult to articulate their reasoning. They appeared to lack confidence in their approaches when discussing solutions as a group and I think that it is therefore important to emphasise and value process over correct answers.

Cassie's Task

The mathematical task "Fill up the Hotel" (source unknown) encouraged students to explore working with the four operations on whole numbers. For this task, pairs of students were given a game board (Figure 3), 20 counters, and two 6-sided dice. The aim of the game is to fill up each of the hotel room numbers (1 to 20) with counters. Students were asked to roll two dice and perform any operation on the two numbers rolled by the dice, utilising the answers to these operations as the hotel room numbers. For example, one student may roll a 2 and a 4 and could make 2+4=6, 4-2=2, 4^2=16, 2^4=16 and so forth. Students were asked to record their solutions and strategies in their journals. The lesson concluded with a discussion of the strategies used.

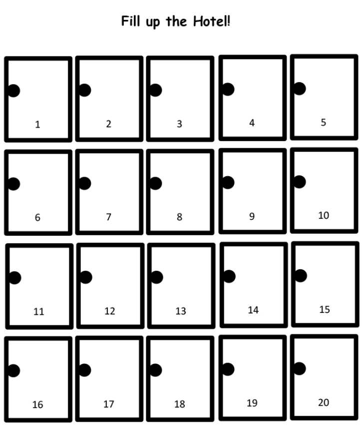


Figure 3. Gameboard for Cassie's task.

By encouraging students to look at ways to take the task further, for example, by adding a third die, changing the hotel room numbers, and estimating how many times it will take them to fill up the entire hotel, they start thinking about more than just doing the formulas and procedures but linking this with their understanding of numbers and operations. This makes the task become more than just a review of the students' procedural knowledge and allows the students to share tactics with one another and assist one another in developing a deeper understanding of their conceptual knowledge.

Conclusion

The process of redesigning a task through a series of iterations as we considered the theory underpinning each pedagogical approach was particularly demanding. However, our understanding of these approaches developed significantly as we adapted aspects of our tasks to explicitly integrate these approaches and during the implementation of our tasks. Our understandings of these learning theories or pedagogical approaches have been strengthened and we will be able to draw on these when designing tasks in the future.

Through the process of designing, adapting and implementing the mathematics tasks, we integrated and evaluated strategies that were designed to support student autonomy, metacognition, and self-regulation. To support student autonomy, we provided students with cognitive autonomy support and this involved allowing students the freedom to use their own strategies to solve problems and providing opportunities for them to discuss and compare multiple strategies. Stephen found that the provision of cognitive autonomy in this way enhanced engagement during his lesson. Chelsea found that allowing students to discuss their strategies provided students with a sense of ownership over their learning.

To promote metacognition, we asked students to articulate their thinking about their thinking. Heidi modeled thinking aloud as she solved her problem and Bethany asked students to annotate their solution pathways with notes that described their thinking. We found that students struggled to articulate their thinking, and instead, tended to describe what they had done. It was evident that for many students, this was an unfamiliar practice and students would benefit by being exposed to the language that helps to articulate their thinking about thinking through teacher modeling. We, as teachers, often model polished solution strategies for our students, and it would be worthwhile to also model our thoughts when we are faced with unfamiliar problems and demonstrate our thinking when we take the wrong path and make mistakes.

We found that self-regulation was promoted through asking students to compare and evaluate solution strategies, and to then refine and apply strategies to similar problems. Interestingly, Scott found that students had difficulty evaluating a single strategy, but when contrasted with another, they were able to discuss and compare these. Asking students to pose problems themselves, and then solve with a revised and improved strategy was another effective way to promote self-regulation. Overall, our task design and implementation was effective in terms of the development of our understanding of teaching for deep learning in mathematics, and in terms of how our students engaged with the tasks.

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FOUR INTERESTING AND USEFUL THEOREMS ABOUT POLYNOMIALS

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There are many theorems involving polynomials that have interesting and useful applications within the typical senior secondary school mathematics curriculum. In this paper four little-known such theorems are stated, proved and illustrated with examples. They provide ideas and procedures that could be used in many types of assessments.

1. Introduction

There are many theorems involving polynomials that have interesting and useful applications within the typical senior secondary school mathematics curriculum. They can be used by teachers to produce specific questions with convenient or 'nice' properties, and can stimulate new and interesting applications of familiar material. The proof of such theorems, with appropriate scaffolding, can be used in analysis tasks or extended-response questions. In this paper, four little-known such theorems are stated, proved and illustrated with examples.

2. Tangents Without Calculus

Students are often taught about polynomial long division in pre-calculus courses. An interesting and often overlooked application of polynomial division is to the calculation of the equation of the tangent to a polynomial curve at a point.

Theorem 1

Let $r(x)$ be the remainder when the polynomial $p(x)$ is divided by $(x-a)^2$. Then $y = r(x)$ is the equation of the tangent to the curve $y = p(x)$ at $x = a$ (Aarao and McKenna 2000, Strickland-Constable 2005, Rabin 2008).

Proof 1

Let $p(x) = (x-a)(x-b)q(x) + r(x),\ \ a,b \in R$ where $r(x)$ is the remainder when $p(x)$ is divided by $(x-a)(x-b)$.

When $p(x)$ is divided by $x-a$ it follows from the remainder theorem that the remainder is $p(a)$. Therefore $r(a) = p(a)$. Likewise, when $p(x)$ is divided by $x-b$ the remainder is $p(b)$ and so $r(b) = p(b)$. It follows that $y = r(x)$ is the secant line through the graph of $y = p(x)$ at the points $(a,\ p(a))$ and $(b,\ p(b))$. If $b \to a$ then the secant line approaches the tangent to $y = p(x)$ at $x = a$.

Proof 2

Let the line $y = mx + c$ be tangent to the curve $y = p(x)$ at $x = a$. Then, following the method of Descartes (Suzuki 2005), the system of equations

$y = p(x)$ and $y = mx + c$

implying $p(x) - (mx + c) = 0$

must have $x = a$ as a double root. Let $p(x) = (x-a)^2 q(x) + r(x)$ where $r(x)$ is the remainder when $p(x)$ is divided by $(x-a)^2$. Then $x = a$ is a double root of

$(x-a)^2 q(x) + r(x) - (mx + c) = 0$.

But the degree of $r(x)$ is less than two since the degree of $(x-a)^2$ is two.

Therefore $r(x) - (mx + c) = 0$

implying $r(x) = mx + c$.

Example 1

Find without using calculus the equation of the tangent to the curve $y = x^2 - 4x + 2$ at the point where $x = -1$.

Method 1

When $x^2 - 4x + 2$ is divided by $(x+1)^2 = x^2 + 2x + 1$ the remainder is $-6x + 1$:

$$\begin{array}{r|l} & \quad\ \ 1 \\ \hline x^2 + 2x + 1 & x^2 - 4x + 2 \\ & \underline{x^2 + 2x + 1} \\ & \quad\ -6x + 1 \end{array}$$

Therefore the equation of the tangent is $y = -6x + 1$.

Method 2 (the method of Descartes)

The coordinates of the tangent point are $(-1,\ 7)$ and so the equation of the tangent is $y-7=m(x+1)$

implying $y=mx+m+7$,

where *m* is a parameter to be determined. In order for $y=mx+m+7$ to be tangent to $y=x^2-4x+2$ at $x=-1$, the system of equations

$y=x^2-4x+2$ and $y=mx+m+7$

implying $x^2-(4+m)x-m-5=0$

must have $x=-1$ as a double root (or equivalently, the discriminant $(4+m)^2+4(m+5)$ is equal to zero). Therefore

$x^2-(4+m)x-m-5\equiv(x+1)^2=x^2+2x+1$·

Comparing coefficients and solving for *m* gives $m=-6$.

Example 2

Find without using calculus the equation of the tangent to the curve $y=x^3-3x^2-x+2$ at the point where $x=1$.

Method 1

When x^3-3x^2-x+2 is divided by $(x-1)^2=x^2-2x+1$ the remainder is $-4x+3$:

$$\begin{array}{r|l} & x-1 \\ \hline x^2-2x+1 & x^3-3x^2-x+2 \\ & \underline{x^3-2x^2+x} \\ & \quad -x^2-2x+2 \\ & \quad \underline{-x^2+2x-1} \\ & \qquad -4x+3 \end{array}$$

Therefore the equation of the tangent is $y=-4x+3$.

Method 2 (the method of Descartes)

The coordinates of the tangent point are $(1,-1)$ and so the equation of the tangent is $y+1=m(x-1)$

implying $y=mx-m-1$,

where m is a parameter to be determined. In order for $y=mx-m-1$ to be tangent to $y = x^3 - 3x^2 - x + 2$ at $x = 1$, the system of equations

$y = x^3 - 3x^2 - x + 2$ and $y = mx - m - 1$

implying $x^3 - 3x^2 - (m+1)x + m + 3 = 0$

must have $x = 1$ as a double root. The requirement that $x = 1$ is a double root means that $(x-1)^2$ is a factor of $x^3 - 3x^2 - (m+1)x + m + 3$. Let the other factor be $(x - r)$.

Then $x^3 - 3x^2 - (m+1)x + m + 3 \equiv (x-1)^2(x-r) = x^3 - (2+r)x^2 + (1+2r)x - r$.

Comparing coefficients gives the system

$3 = 2 + r$, $-(m+1) = 1 + 2r$, $m + 3 = -r$.

Solving this system gives $r = 1$ and $m = -4$.

Note that $r = 1$ shows that $x = 1$ is in fact a triple root of the system and so the tangent intersects the cubic curve exactly once. Therefore the point where $x = 1$ must be the point of inflection of the curve. This observation is confirmed by noting that the solution to $f''(x) = 0$ is $x = 1$.

Aside: The tangent to the point of inflection of a cubic curve is the only tangent to intersect the curve exactly once.

Proof: Let the line $y = mx + c$ be tangent to the cubic curve $y = \alpha x^3 + \beta x^2 + \delta x + \rho$ at the point where $x = k$. Then the system of equations

$y = \alpha x^3 + \beta x^2 + \delta x + \rho$ and $y = mx + c$

implying $\alpha x^3 + \beta x^2 + (\delta - m)x + \rho - c = 0$

must have $x = k$ as a triple root if the tangent intersects the curve exactly once. Therefore

$\alpha x^3 + \beta x^2 + (\delta - m)x + \rho - c \equiv \alpha(x-k)^3 = \alpha(x^3 - 3kx^2 + 3k^2x - k^3)$

and comparing coefficients gives $k = \dfrac{-\beta}{3\alpha}$. The solution to $\dfrac{d^2y}{dx^2} = 0$ is $k = \dfrac{-\beta}{3\alpha}$ and so k is the x-coordinate of the point of inflection of $y = \alpha x^3 + \beta x^2 + \delta x + \rho$.

Tangents that intersect a curve exactly once are called Descartes lines. It can be shown that a Descartes line must pass through a point of inflection and that a polynomial of odd degree $2n + 1$ has n Descartes lines (Barnier 2007).

Example 3

Find without using calculus the equation of the tangent to the curve $y = x^4 + 4x^3 - 44x^2 - 96x$ at the point where $x = -1$.

Method 1

When $x^4 + 4x^3 - 44x^2 - 96x$ is divided by $(x+1)^2$ the remainder is 49. Therefore the equation of the tangent is $y = 49$ (showing that the point where $x = -1$ is a stationary point).

Method 2 (the method of Descartes)

The coordinates of the tangent point are $(-1, 49)$ and so the equation of the tangent is $y = mx + m + 49$. It follows that $(x+1)^2$ is a factor of

$x^4 + 4x^3 - 44x^2 - (96+m)x - m - 49$

implying $x^4 + 4x^3 - 44x^2 - (96+m)x - m - 49 \equiv (x+1)^2(x^2 + bx + c)$.

Comparing coefficients gives the system

$2 = b$, $-45 = 2b + c$, $-96 - m = b + 2c$, $-49 = c$

and solving for *m* gives $m = 0$.

The tangent also intersects the curve at the points where $x^2 + 2x - 49 = 0$.

3. 'Nice' Polynomials

It can be useful when writing a calculus-based CAS-free assessment to be able to produce a 'nice' polynomial with rational (ideally small integer) roots such that its first derivative also has rational roots. Such polynomials are called *Q*-nice.

If $p(x)$ is a *Q*-nice polynomial then the transformed polynomial $\alpha p(\beta(x-\delta))$ is also *Q*-nice provided $\alpha, \beta, \delta \in Q$. All *Q*-nice polynomials can therefore be generated from a set of standard types.

3.1 'Nice' Cubics

Q-nice cubics always exist in the case of a root repeated two or three times:

If $f(x) = (x-a)^2(x-b)$ then $f'(x) = (x-a)(3x-2b-a)$ and the roots of $f(x)$ and $f'(x)$ are obviously rational if $a, b \in Q$.

The remaining cases of three distinct real roots and one real root are therefore discussed.

3.1.1 'Nice' Cubics With Three Distinct Real Roots

Theorem 2

Let *p*, *q*, *r*, *s* form an arithmetic progression and either $p, q, r, s \in Q$ or $pr, qs \in Q$. Then the cubic function $f(x) = x(x-pr)(x-qs)$ is *Q*-nice (Chapple 1960).

Proof

Let $f(x)=x(x-a)(x-b)=x^3-(a+b)x^2+abx$, $a,b\in Q$. .... (1)

Then $f'(x)=3x^2-2(a+b)x+ab$. .... (2)

If $f(x)$ is Q-nice then

$f'(x)=3(x-c)(x-d)=3x^2-3(c+d)x+3cd$, $c,d\in Q$. .... (3)

Compare coefficients in (2) and (3):

$2(a+b)=3(c+d)$ and $ab=3cd$ .... (4)

implying $2\left(\frac{3cd}{b}+b\right)=3(c+d)$

implying $2b^2-3(c+d)b+6cd=0$. .... (5)

The discriminant of the quadratic in (5) is a perfect square since b is rational. Therefore:

$9(c+d)^2-48cd=n^2$, $n\in Z$ .... (6)

implying $9d^2-30cd+9c^2=n^2$.

Complete the square:

$(3d-5c)^2-16c^2=n^2$

implying $n^2+(4c)^2=(3d-5c)^2$.

Therefore the triplet $(n,4c,3d-5c)$ has the form of a Pythagorean triad. It follows from the standard parameterisation of Pythagorean triads (Stillwell 1998 p 114) that

$n=\frac{u^2-v^2}{2}w$, $4c=uvw$, $3d-5c=\frac{u^2+v^2}{2}w$, $u^2,v^2,uv,w\in Q$. .... (7)

From (7):

$c=\frac{uvw}{4}$. .... (8)

$d=\frac{u^2+v^2}{6}w+\frac{5c}{3}=\frac{u^2+v^2}{6}w+\frac{5uv}{12}w$

$=\frac{w(2u^2+2v^2+5uv)}{12}$ .... (9)

$=\frac{w(2u+v)(u+2v)}{12}$. .... (10)

It follows from (5) and (6) that $b=\frac{3c+3d\pm n}{4}$. Take $b=\frac{3c+3d+n}{4}$ (choosing the minus sign solution gives the same results) and substitute from (8), (9) and (7):

$$b = \frac{w}{4}\left(\frac{3uv}{4} + \frac{3(2u^2 + 2v^2 + 5uv)}{12} + \frac{u^2 - v^2}{2}\right)$$

$$= \frac{wu(2v+u)}{4} \quad \text{.... (11)}$$

From (4): $a = \frac{3d}{b}$.

Substitute from (8), (10) and (11):

$$a = \frac{3wv(2u+v)}{4}. \quad \text{.... (12)}$$

Since Q-nice cubics can be transformed into other Q-nice cubics, the simplifying substitution $w = 12$ can be made in (8), (10), (11) and (12):

$a = 3v(2u+v)$ and $b = 3u(2v+u)$. .... (13)

$c = 3uv$ and $d = (2u+v)(u+2v)$. .... (14)

The 'factors' $p = 3v$, $q = 2v+u$, $r = v+2u$, $s = 3u$ of a and b form an obvious arithmetic progression (with common difference $u - v$). Then from (1), (3), (13) and (14):

$f(x) = x(x - pr)(x - qs)$. .... (15)

$f'(x) = 3(x - \frac{ps}{3})(x - qr) = (3x - ps)(x - qr)$. .... (16)

Other methods for producing equivalent *Q*-nice cubics can be found in Galvin (1990), Buddenhagen, Ford and May (1992) and Gupta and Szymanski (2010).

Example 4

Let $p = 1$, $q = 2$, $r = 3$ and $s = 4$ in (15).

Then $f(x) = x(x-3)(x-8)$ is *Q*-nice.

$f(x+3) = x(x+3)(x-5)$ is Q-nice.

$f(x+2) = g(x) = (x+2)(x-1)(x-6)$ is Q-nice.

$g(2x) = (2x+2)(2x-1)(2x-6) = 4(x+1)(2x-1)(x-3)$ is Q-nice.

Example 5

Let $p = \sqrt{3}$, $q = 2\sqrt{3}$, $r = 3\sqrt{3}$ and $s = 4\sqrt{3}$ in (15).

Then $f(x) = x(x-9)(x-24)$ is *Q*-nice. In fact, the roots of $f'(x) = (3x-12)(x-18)$ are all integers and so $x(x-9)(x-24)$ is *Z*-nice. *Z*-nice cubics can be produced using the following Corollary.

Corollary 1

Let *p*, *q*, *r*, *s* form an arithmetic progression and either each number is an integer with

p divisible by 3 or each number has an integer and $\sqrt{3}$ as its only 'factors'. Then the cubic function $f(x) = x(x - pr)(x - qs)$ is Z-nice.

Corollary 1 readily follows from (16). 'Niceness preserving' transformations can be applied to get more convenient Z-nice cubics with smaller roots, noting that the dilation $f(x) \to f(kx)$ is only 'Z-nice preserving' if k is a common divisor of the roots of $f'(x)$. General formulas that give Z-nice cubics are given in Evard (2004).

Example 6

Let $p = 6$, $q = 5$, $r = 4$ and $s = 3$ in (15).

Then $f(x) = x(x - 24)(x - 15)$ is Z-nice.

$f(x + 20) = g(x) = (x + 20)(x - 4)(x + 5)$ is Z-nice.

The derivative of $g(2x) = (2x + 20)(2x - 4)(2x + 5) = 4(x + 10)(x - 2)(2x + 5)$ has integer roots.

The derivative of the function in (15) is $3x^2 - 2(pr + qs)x + pqrs$. Comparing this with (16) shows that $2(pr + qs) = ps + 3qr$. This identity is easily proved to be true for any arithmetic progression:

Let the terms of the arithmetic progression be p, $q = p + d$, $r = p + 2d$, $s = p + 3d$.

RHS $= ps + 3qr = p(p + 3d) + 3(p + d)(p + 2d) = 4p^2 + 12pd + 6d^2$.

LHS $= 2(pr + qs) = 2(p + d)(p + 3d) + 2p(p + 2d) = 4p^2 + 12pd + 6d^2 =$ RHS.

The existence of similar identities and generalisations for terms of an arbitrary arithmetic progression are discussed in MacDougall (1995).

3.1.2 'Nice' Cubics With One Real Root

Theorem 3

The cubic function $f(x) = x\left(x^2 - \frac{3}{2}(\alpha + \beta)x + 3\alpha\beta\right)$ has one real root and is Q-nice when either $\frac{\beta}{3} < \alpha < 3\beta$ and $\alpha, \beta \in Z^+$ or $3\beta < \alpha < \frac{\beta}{3}$ and $\alpha, \beta \in Z^-$.

Special case: $f(x) = (x - \alpha)^3 + \alpha^3$ when $\alpha = \beta$.

Proof

Let $f(x) = x(x^2 + bx + c)$ where the quadratic factor is irreducible and so

$b^2 - 4c < 0$. .... (17)

Then $f'(x) = 3x^2 + 2bx + c$ . .... (18)

If $f(x)$ is Q-nice then

$f'(x) = 3(x-\alpha)(x-\beta) = 3x^2 - 3(\alpha+\beta)x + 3\alpha\beta$, $\alpha, \beta \in Q$ . .... (19)

Compare coefficients in (18) and (19):

$2b = -3(\alpha+\beta)$ and $c = 3\alpha\beta$ . .... (20)

Substitute (20) into (17) and simplify:

$3\alpha^2 - 10\alpha\beta + 3\beta^2 < 0$

implying $(3\alpha-\beta)(\alpha-3\beta) < 0$.

There are two cases:

1. $\alpha > \frac{\beta}{3}$ and $\alpha < 3\beta$, that is, $\frac{\beta}{3} < \alpha < 3\beta$, which is only satisfied if $\beta > 0$.

2: $\alpha < \frac{\beta}{3}$ and $\alpha > 3\beta$, that is, $3\beta < \alpha < \frac{\beta}{3}$, which is only satisfied if $\beta < 0$.

Special case: The derivative function in (19) has exactly one root when the discriminant is equal to zero:

$9(\alpha+\beta)^2 - 36\alpha\beta = 0$

implying $\alpha = \beta$

implying $f(x) = x(x^2 - 3\alpha x + 3\alpha^2) = (x-\alpha)^3 + \alpha^3$.

Example 7

Let $\alpha = 4$ and $\beta = 2$ in theorem 3. Then $f(x) = x(x^2 - 9x + 24)$ has one real root and is *Q*-nice.

3.2 'Nice' Quartics

Q-nice quartics always exist in the cases of a root repeated three or four times or two double roots:

If $f(x) = (x-a)^3(x-b)$ then $f'(x) = (x-a)^2(4x-3b-a)$.

If $f(x) = (x-a)^2(x-b)^2$ then $f'(x) = 2(x-a)(x-b)(2x-b-a)$.

In both cases the roots of $f(x)$ and $f'(x)$ are obviously rational if $a, b \in Q$.

Richardson (1960) gave the following method for producing *Q*-nice quartics with four distinct and real roots:

Let *p*, *q*, *r* be consecutive numbers. Then

$f(x) = x(x-2q)(x-pr)(x-[2q+pr])$ is Q-nice and
$f'(x) = 2(x-p)(x-qr)(2x-[2q+pr])$.

Since $(2q)^2 + (pr)^2 = 4(p+1)^2 + p^2(p+2)^2 = (p^2+2p+2)^2$ is a perfect square, the roots $2q$ and pr are the first two elements of a Pythagorean triad. This observation is generalised in the following theorem.

Theorem 4

Let $b, c \in Q$ and $b^2 + c^2$ be a perfect square. Then the quartic function $f(x) = x(x-b)(x-c)(x-[b+c])$ is *Q*-nice. (Richardson 1960).

Proof

Let three of the roots of a quartic f(x) be 0, b, c and choose

$$f(x) = x(x-b)(x-c)(x-[b+c] = (x^2-[b+c]x)(x^2-[b+c]x+bc) \ .$$

The choice of $b+c$ for the fourth root is made so that $f(x)$ will have two quadratic factors that each start with the same two terms and hence produce the simple factor $(2x-[b+c])$ in $f'(x)$:

$$f'(x) = (2x-[b+c])(2x^2-2[b+c]x+bc)$$

which will have rational roots if $2x^2-2[b+c]x+bc$ has rational roots. That is, if the discriminant $4(b+c)^2-8bc = 4(b^2+c^2)$ is a perfect square, that is, if b^2+c^2 is a perfect square.

Note that

$$f\left(x+\frac{b+c}{2}\right) = \left(x+\frac{(b+c)}{2}\right)\left(x-\frac{(b-c)}{2}\right)\left(x+\frac{(b-c)}{2}\right)\left(x-\frac{(b+c)}{2}\right)$$

$$= \left(x^2-\frac{(b+c)^2}{4}\right)\left(x^2-\frac{(b-c)^2}{4}\right)$$

$$= x^4 - \frac{1}{2}(b^2+c^2)x^2 + \frac{1}{16}(b^2-c^2)^2 \qquad \text{.... (21)}$$

is a symmetric quartic (since it contains no odd-powered terms). Caldwell (1990) gives a method for producing Z-nice symmetric quartics that are equivalent to (21). He also gives the first five known examples of non-symmetric Z-nice quartics with four distinct and real roots (Caldwell 1990). Using an efficient computational procedure, Evard (2004) has found 358 other examples of non-symmetric Z-nice quartics, the 'smallest' one being $x(x-50)(x-176)(x-330)$ (not particularly suited for classroom work). A general formula that gives Q-nice non-symmetric quartics remains an open question.

Methods for finding rational-derived quartics (all derivatives have rational roots) with a double root and two other distinct roots are given in Carroll (1989), Galvin and MacDougall (1994) and Buchholz and Kelly (1995).

Q-nice and rational-derived polynomials in general are studied and classified in Buchholz and MacDougall (2000), Flynn (2001) and Evard (2004).

4. The Cayley-Hamilton Theorem

The Cayley-Hamilton theorem is an important and versatile theorem of linear algebra. It is useful for finding the inverse of a non-singular matrix and for finding powers of a square matrix.

4.1 Definition

If M is an $n \times n$ matrix, then the characteristic equation of M is $\det(M - \lambda I) = 0$. The solutions to the characteristic equation are the eignenvalues of M and have many important applications (HREF2).

4.2 Cayley-Hamilton Theorem

If M is an $n \times n$ matrix, O is the zero $n \times n$ matrix and $\phi(\lambda) = 0$ is the characteristic equation of M, then $\phi(M) = O$. In other words, M satisfies its own characteristic equation. (HREF3)

Example 8

Consider $A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 3 & -1 \\ -1 & -2 & 1 \end{bmatrix}$. Then $\det(A - \lambda I) = 0$

implying $\det \begin{bmatrix} 4-\lambda & 1 & -2 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 1-\lambda \end{bmatrix} = \lambda^3 - 8\lambda^2 + 14\lambda - 2 = 0$

implying $A^3 - 8A^2 + 14A - 2I = O$

implying $A^2 - 8A + 14I - 2A^{-1} = O$

implying $A^{-1} = \frac{1}{2}(A^2 - 8A + 14I)$.

Example 9

Consider $A = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix}$. Then $\det(A - \lambda I) = 0$

implying $\det\begin{bmatrix}3-\lambda & -4 \\ 1 & -2-\lambda\end{bmatrix} = \lambda^2 - \lambda - 2 = 0$

implying $A^2 - A - 2I = O$. Therefore:

$$
\begin{aligned}
A^2 & & &= A + 2I\,, \\
A^3 &= A(A+2I) = A^2 + 2A = (A+2I) + 2A & &= 3A + 2I\,, \\
A^4 &= A(3A+2I) = 3A^2 + 2A = 3(A+2I) + 2A & &= 5A + 6I\,, \\
A^5 &= A(5A+6I) = 5A^2 + 6A = 5(A+2I) + 6A & &= 11A + 10I \;\ldots.
\end{aligned}
$$

The coefficients of *A* and *I* in the above expressions for A^n form a sequence of numbers that can be generated from a recurrence relation. Let s_n and t_n be the coefficients of *A* and *I* respectively in the expressions for A^n. Then:

$s_n = s_{n-1} + t_{n-1}$ and $t_n = 2s_{n-1}$, $s_1 = 1$ and $t_1 = 0$

from which it follows that:

- $s_n = s_{n-1} + 2s_{n-2}$, $s_1 = 1$ and $s_2 = 3$

implying $s_n = \frac{2}{3}(2)^n + \frac{1}{3}(-1)^n$.

- $t_n = t_{n-1} + 2t_{n-2}$, $t_1 = 0$ and $t_2 = 2$

implying $t_n = \frac{2}{3}(2)^n - \frac{2}{3}(-1)^n$.

5. Average of the Roots of a Polynomial

The average of the roots of a polynomial $p(x)$ is the same as the average of the roots of the derivative $p'(x)$. (Kalman 2009 p 50)

Proof

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.

Using *Vieta's formulas* (HREF1): Sum of the roots of $p(x) = \dfrac{-a_{n-1}}{a_n}$.

Therefore the average of the roots of $p(x) = \dfrac{-a_{n-1}}{na_n}$.

$p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1$.

Using *Vieta's formulas*: Sum of the roots of $p'(x) = \dfrac{-(n-1)a_{n-1}}{na_n}$.

Therefore the average of the roots of $p'(x) = \dfrac{-(n-1)a_{n-1}}{na_n(n-1)} = \dfrac{-a_{n-1}}{na_n}$.

Example 10

Consider the Z-nice cubic $p(x) = x(x-9)(x-24)$. The average of the roots of $p(x)$

, $p'(x) = (3x-12)(x-18)$ and $p''(x) = 6x - 66$ is 1 in each case.

Proof by induction of Vieta's formula:

Step 1: True for $n = 2$.

Let $f(x) = a_2x^2 + a_1x + a_0$ have roots β_2 and β_1. Then

$f(x) = a_2(x-\beta_2)(x-\beta_1) = a_2x^2 - a_2(\beta_2+\beta_1)x - a_2\beta_2\beta_1$.

Compare the coefficients of x: $a_1 = -a_2(\beta_2+\beta_1) \Rightarrow \beta_2 + \beta_1 = \dfrac{-a_1}{a_2}$.

Step 2 (**inductive hypothesis**): Assume true for $n = k$.

Let $f(x) = a_kx^k + a_{k-1}x^{k-1} + \ldots + a_1x + a_0$ have roots β_k, β_{k-1}, ..., β_1. Then

$f(x) = a_k(x-\beta_k)(x-\beta_{k-1})\cdots(x-\beta_1)$ and $\beta_k + \beta_{k-1} + \ldots + \beta_1 = \dfrac{-a_{k-1}}{a_k}$.

Step 3: Show true for $n = k+1$.

Let $f(x) = a_{k+1}x^{k+1} + a_kx^k + a_{k-1}x^{k-1} \ldots + a_1x + a_0$ have roots β_{k+1}, β_k, β_{k-1}, ..., β_1. Then

$f(x) = a_{k+1}(x-\beta_{k+1})(x-\beta_k)(x-\beta_{k-1})\cdots(x-\beta_1)$

$= \dfrac{a_{k+1}}{a_k}(x-\beta_{k+1})(a_kx^k + a_{k-1}x^{k-1} + \ldots + a_1x + a_0)$.

Substitute $\beta_k + \beta_{k-1} + \ldots + \beta_1 = \dfrac{-a_{k-1}}{a_k}$

implying $a_{k-1} = -a_k(\beta_k + \beta_{k-1} + \ldots + \beta_1)$

from Step 2 (the inductive hypothesis):

$f(x) = \dfrac{a_{k+1}}{a_k}(x-\beta_{k+1})(a_kx^k - a_k[\beta_k + \beta_{k-1} + \cdots + \beta_1])x^{k-1} + \cdots)$.

Expand:

$f(x) = a_{k+1}x^{k+1} + \left(\dfrac{a_{k+1}}{a_k}\right)(-a_k[\beta_k + \beta_{k-1} + \ldots + \beta_1])x^k - \left(\dfrac{a_{k+1}}{a_k}\beta_{k+1}\right)a_kx^k + \ldots$

$= a_{k+1}x^{k+1} - a_{k+1}(\beta_k + \beta_{k-1} + \cdots + \beta_1)x^k - a_{k+1}\beta_{k+1}x^k + \cdots$

$= a_{k+1}x^{k+1} - a_{k+1}(\beta_{k+1} + \beta_k + \beta_{k-1} + \ldots + \beta_1)x^k + \ldots$.

But $f(x) = a_{k+1}x^{k+1} + a_kx^k + a_{k-1}x^{k-1} \ldots + a_1x + a_0$.

Compare the coefficient of x^k in each expression for $f(x)$:

$a_k = -a_{k+1}(\beta_{k+1} + \beta_k + \alpha_{k-1} + \ldots + \beta_1)$

implying $\beta_{k+1} + \beta_k + \beta_{k-1} + \ldots + \beta_1 = \dfrac{-a_k}{a_{k+1}}$.

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A PRIME TIME WITH M*ATHEMATICA*

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In this paper the number functionality of a CAS such as Mathematica is used to provide various sample computations that could be used to support broader exploration of prime numbers in the secondary curriculum, in particular where larger numbers are involved.

Introduction

The last few years have been a 'big' time for discovering new prime numbers and results about prime numbers. In late January 2013 a new largest known prime number $2^{57885161}-1$ was announced by the Great Internet Mersenne Prime Search (GIMP) project (HREF1). The largest currently known factorial prime is 150209! + 1 was identified in late 2011. Technology is a natural tool for working mathematically in this context.

In April 2013 a Chinese-American mathematician, Yitang Zhang, announced a proof that there are infinitely many pairs of consecutive primes with a gap at most 70 million (this result has since been improved on). This proof is the first to establish the existence of a finite bound for such gaps. The significance of this result, and why many mathematicians *believe* that the twin primes conjecture 'there are infinitely many primes p such that $p+2$ is also prime' is true*, even though there is yet no proof* is discussed nicely in the article by Ellenberg (HREF2). Prime numbers are an active area of mathematical research with several well-known conjectures for investigation:

- every even natural number greater than two can be expressed as a sum of two primes (Goldbach's Conjecture);

- there are infinitely many primes p such that $p+2$ is also prime (the twin prime conjecture);
- there is an odd perfect number (a perfect number is a natural number that is the sum of its positive divisors excluding itself)
- the abc conjecture (HREF3)

Prime Numbers in the Curriculum

The study of prime numbers has typically been part of the school mathematics curriculum in the upper primary and early secondary years, most recently the Australian Curriculum: Mathematics Foundation – Year 10:

Year 5

Identify and describe factors and multiples of whole numbers and use them to solve problems (ACMNA098)

Year 6

Identify and describe properties of prime, composite, square and triangular numbers (ACMNA122)

Year 7

Investigate index notation and represent whole numbers as products of powers of prime numbers (ACMNA149)

(HREF4).

Topic content usually includes divisors and multiples, factors of natural numbers, prime and composite numbers, factor sets and factor trees, classification of types of numbers, their representations and the fundamental theorem of arithmetic. In later secondary years students may also study prime numbers as part of enrichment and extension of the curriculum in the area of number theory, or as a component of school-based assessment, such as investigation of the Rivest-Shamir-Adleman (RSA) public key cryptosystem algorithm based on the difficulty of factorizing large numbers. Students will likely have used the Sieve or Eratosthenes on a grid of the numbers from 1 to 100 as shown in Figure 1. The Sieve of Eratosthenes is efficient for finding 'small' prime numbers, and for a given natural number, n, tests potential divisors up to $\sqrt{n}$.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1. Sieve of Eratosthenes for 1 to 100 grid.

Students do not usually get to work with large prime numbers (ask any student what is the largest prime number that they know for sure) or indeed, large lists of prime numbers. This is certainly likely to be the case when the relevant computations for prime number identification are carried out by hand or using a scientific calculator. CAS such as *Mathematica* have specific number theoretic functionality that can be used to support a more detailed study of prime numbers. There are also related Demonstrations (dynamic pre-developed *Mathematica* files) available from Wolfram Research that use this functionality to illustrate aspects of mathematics such as divisibility networks, the Sieve or Eratosthenes, prime factorisation or the distribution of primes.

A Few Basics

In the following discussion the word 'number' is taken to refer to a natural number, that is an element of the set $N = \{1, 2, 3 \ldots\}$. The set of prime numbers $P = \{2, 3, 5, 7 \ldots\}$ is a proper subset of N. A number is said to be a prime if it has exactly two distinct factors, 1 and itself, while a number with more than two distinct factors is said to be composite. A special type of composite number is a factorial where 'n factorial' is $n! = n \times (n-1) \times (n-2) \times (n-3) \ldots \times 3 \times 2 \times 1$, for example $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

The number 17 is prime as its only factors are 1 and 17 (exactly two distinct factors); while 24 is composite as it has eight distinct factors

{1, 2, 3, 4, 6, 8, 12, 24}. The number 1 is a special case—it is neither prime nor composite as it has only one factor—itself (this is the only number that is neither prime nor composite). Prime numbers are important in mathematics because they can be used to build up (compose) and uniquely represent (up to order) other numbers using only multiplication, for example 2013 = 3 × 11 × 61.

It is not immediately clear whether an arbitrary number such as 1 234 567 891 011 is prime or not. Some interesting related mathematical questions are: How can one identify whether a number is prime or not? Are prime numbers easy to find or not? Is there a rule for generating all prime numbers? How are prime numbers used to build up composite numbers? How many prime numbers are there? Where do they occur in *N*?

Some *Mathematica* Number Functionality

Mathematica has various number theoretic functionalities.

Finding divisors

```
Divisors[{48, 101, 11111}]

{{1, 2, 3, 4, 6, 8, 12, 16, 24, 48}, {1, 101}, {1, 41, 271, 11111}}
```

Identifying prime numbers

PrimeQ[n]tests whether n is prime or not

```
PrimeQ[1234567]

False
```

Selecting all the primes from within a given set

```
Select[Table[n, {n, 1, 100}], PrimeQ]

{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97}
```

Finding the n^{th} prime number

Prime[**n**] gives the n^{th} prime number

```
{Prime[10], Prime[2013]}

{29, 17491}
```

Generating a list of the first n primes

```
Table[{n, Prime[n]}, {n, 1, 100}]

{{1, 2}, {2, 3}, {3, 5}, {4, 7}, {5, 11}, {6, 13}, {7, 17}, {8, 19}, {9, 23},
 {10, 29}, {11, 31}, {12, 37}, {13, 41}, {14, 43}, {15, 47}, {16, 53},
 {17, 59}, {18, 61}, {19, 67}, {20, 71}, {21, 73}, {22, 79}, {23, 83},
 {24, 89}, {25, 97}, {26, 101}, {27, 103}, {28, 107}, {29, 109}, {30, 113},
 {31, 127}, {32, 131}, {33, 137}, {34, 139}, {35, 149}, {36, 151}, {37, 157},
 {38, 163}, {39, 167}, {40, 173}, {41, 179}, {42, 181}, {43, 191}, {44, 193},
 {45, 197}, {46, 199}, {47, 211}, {48, 223}, {49, 227}, {50, 229}, {51, 233},
 {52, 239}, {53, 241}, {54, 251}, {55, 257}, {56, 263}, {57, 269}, {58, 271},
 {59, 277}, {60, 281}, {61, 283}, {62, 293}, {63, 307}, {64, 311}, {65, 313},
 {66, 317}, {67, 331}, {68, 337}, {69, 347}, {70, 349}, {71, 353}, {72, 359},
 {73, 367}, {74, 373}, {75, 379}, {76, 383}, {77, 389}, {78, 397}, {79, 401},
 {80, 409}, {81, 419}, {82, 421}, {83, 431}, {84, 433}, {85, 439}, {86, 443},
 {87, 449}, {88, 457}, {89, 461}, {90, 463}, {91, 467}, {92, 479}, {93, 487},
 {94, 491}, {95, 499}, {96, 503}, {97, 509}, {98, 521}, {99, 523}, {100, 541}}
```

Finding the number of primes less than a given number

```
PrimePi[100]

25
```

PrimePi[**n**]can be used to find numbers of primes in an interval, for example between 1 000 and 10 000 and can be approximated by $n/\log_e(n)$:

```
PrimePi[10000] - PrimePi[1000]

1061
```

As the size of prime numbers increases so does the gap between consecutive primes. For $n > 1$ it is known that there is at least one prime between n and $2n$. For $n > 3$, $\{n! + 2,\ n! + 3 \ldots\ n! + n\}$ forms a sequence of $n - 1$ consecutive composite numbers.

FactorInteger[n] computes the prime factors of n and their power in its prime factorisation, f or example:

```
FactorInteger[9864]

{{2, 3}, {3, 2}, {137, 1}}
```

Thus the prime factorisation of 9864, that is, its representation as a product of powers of primes is $2^3 \times 3^2 \times 137$. The following is a short program in *Mathematica* which defines a new function **Factorise**[**n**] that does the re-writing:

```
Factorise[n_] := If[Length[FactorInteger[n]] > 1,
  Cross @@ (Superscript @@@ FactorInteger[n]),
  First[Superscript @@@ FactorInteger[n]]]

Factorise[9864]

2^3 × 3^2 × 137^1
```

The corresponding table for n from 1 to 100 shows that at most four prime factors are used for a given number and that the powers of these are generally small.

Finding Primes

There is no largest prime so *P* is an infinite set. There are several proofs - some of them such as Euclid's proof provide a method for finding a 'new' prime. A particular type of prime number is a factorial prime. Factorials can be used to illustrate how one can find some 'new' primes. Suppose one has already identified all the primes less than or equal to n. Clearly $n!$ is composite, but what about $n!+1$? None of the numbers 2, 3 ... n can be a factor of $n!+1$, since there will be a remainder of 1 on division. Nor can they be a factor of any other number that is a factor of $n!+1$. It may be the case that $n!+1$ is prime, for example if $n = 3$ then 3! + 1 = 7 which is prime. On the other hand $n!+1$ may not be prime, for example, if $n = 4$ then 4! + 1 = 25. Here 5 is a factor of 25 and is a 'new' prime. If a factor of $n!+1$ which has been found is not prime then the same reasoning can be applied to this number and so on, however this process must stop since the factors found in this sequence are getting smaller and all such factors must be greater than n. In the worst possible scenario it will turn out that $n!+1$ is a 'new' prime as is the case for $n=4$. However, as indicated in the following computations, when $n=10$ and hence 10! + 1 = 3628801 we do not have a prime number, but the 'new' primes found are 11 and 329 891.

```
{10 ! + 1, PrimeQ[10 ! + 1], Factorise[10 ! + 1]}

{3 628 801, False, 11^1 × 329 891^1}
```

The preceding process seems to be a lot of work for a few 'new' primes, one might wish for a simple generating formula. There is no known simple polynomial function for generating the n^{th} prime number. However there are a several low-order polynomial functions that generate prime numbers for initial subsets of *N*, perhaps one of the best known of these is $p(n)=n^2-n+41$ (Euler) which generates 40 distinct primes on {0, 1, 2 ... 40}. There is an even simpler (but not quite so good) function $p(n)=2n^2+29$ (Legendre) which generates 29 distinct primes for {0, 1, 2 ... 28}.

```
Table[2 n^2 + 29, {n, 0, 30}]

{29, 31, 37, 47, 61, 79, 101, 127, 157, 191, 229,
 271, 317, 367, 421, 479, 541, 607, 677, 751, 829, 911,
 997, 1087, 1181, 1279, 1381, 1487, 1597, 1711, 1829}
```

As can be seen from the following computations, the last two elements in the list are not prime.

```
PrimeQ[Table[2 n^2 + 29, {n, 0, 30}]]

{True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, False, False}
```

Students could investigate the efficacy of various linear, quadratic and other polynomial functions for generating distinct primes on initial subsets of *N*. There are quite a few such quadratic functions, what about linear functions? In 2010 Jens Anderson found that $l(n) = 5283234035979900n + 43142746595714191$ is prime for all n from 0 to 25. Conversely, some functions 'avoid' primes, for example $f(n) = n^6 + 1091$ does not generate primes for $n! + 1$ to 3095. What are some other 'prime-avoiding' polynomial functions?

These and related computations provide teachers and students with the opportunity to readily generate a range of examples and counter-examples that can be used as a basis for formulating and testing conjectures. They can be naturally complemented by proofs as applicable, in particular where these are of a constructive nature. A range of results and proofs involving prime numbers can be found at the Australian Mathematical Sciences Institute website (HREF5).

Conclusion

Most students only get to work with small prime numbers, and small sets of prime numbers, so it's not surprising if they wonder what all the fuss is about and, well, could we do without it? There is a much richer set of examples that can be accessed with the number functionality of a CAS such as *Mathematica*. This can be used to further explore related properties of, and conjectures about prime numbers.

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WHY CUBIC POLYNOMIALS?

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Why include study of cubic polynomial functions in the senior secondary curriculum? Why study what we do and the way we do it? This paper discusses some related considerations.

Introduction

The study of cubic polynomial functions of a single real variable is typically introduced in Year 11 as a generalization of work on linear functions and quadratic functions across Years 7 - 10, and then extended to include calculus. Initially this generally includes definition and representation in the form $p: R \rightarrow R, p(x) = ax^3 + bx^2 + cx + d, a \neq 0$, perhaps with use of index notation for the coefficients and evaluation of $p(x)$ for various integer and occasional fraction or decimal value for x using function notation. This is then typically followed by algebraic manipulation for computation of scalar multiple, addition, subtraction, multiplication (expanding) and division by a linear term. Finally there is some technical application of these skills to re-expression with respect to a linear term leading to the remainder and factor theorems and factorization with respect to examples for which a rational (usually integer) root exists; solving selected equations where such a root exists and sketching related graphs. Some modelling applications might then be considered, more so when some calculus (differentiation) has subsequently been covered. Some work may also be done using simultaneous linear equations or perhaps difference equations to determine coefficients given suitable sets of information. Anti-differentiation and possibly also some integration and standard applications would be covered later in the year.

Much of this work is done with carefully selected examples that are amenable to working through with by hand calculation and algebraic manipulation that is deemed 'accessible' to most students doing this course of study. A key aspect of mathematics is being able to recognize and solve *general* classes of problems, so it is natural to inquire as to the extent to which an approach similar to that described above supports this. Consider the case where the coefficients of a cubic polynomial function $p(x) = ax^3 + bx^2 + cx + d$ are randomly selected integers where a is non-zero, or possibly decimal approximate values obtained from empirical data in a modelling context.

To what extent are the techniques that students might typically study and practice likely to be of assistance in analyzing the behavior of this type of function in general and any applications? For example, if $a = 2, b = 4, c = -6, d = 7$ the corresponding graph is shown in Figure 1:

```
f[x_] := 2 x^3 + 4 x^2 - 6 x + 7

Plot[f[x], {x, -5, 4}, AxesLabel → {x, f}]
```

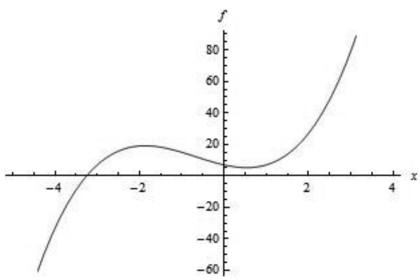

Figure 1. Graph of a cubic polynomial function

The exact value of the single irrational real root that corresponds to the horizontal axis intercept can be obtained from the cubic formula, which in the above case leads to:

$$x \to \frac{1}{3}\left(-2 - \frac{13 \times 2^{2/3}}{\left(329 - 3\sqrt{8121}\right)^{1/3}} - \frac{\left(329 - 3\sqrt{8121}\right)^{1/3}}{2^{2/3}}\right)$$

The derivation of this formula is not covered as a standard component of senior secondary courses; indeed it is a bit involved. Practically a numerical method is typically used. A rational approximation to this root is -3.253. The locations of the stationary points are at:

$$x \to \frac{1}{3}\left(-2 - \sqrt{13}\right) \text{ and } x \to \frac{1}{3}\left(-2 + \sqrt{13}\right)$$

Rational approximations for these are -1.869 and 0.535 respectively.

Of course we can *devise* a function *f* for example, $f(x) = (x+4)(x^2-4x+6)$ which we know will have the same graphical features as f but with a single integer valued axis intercept at $x = 4$, and present it to students for analysis. We do this *because* we need it to be accessible for by hand work to reasonably take place. However given many of the examples and exercises students are often asked to work on as part of customary practice, they could be forgiven for forming the view that for 'any' cubic polynomial function they will 'likely' come across, they will be able to use the factor theorem to find an integer root unless: they are unlucky; they haven't tried hard enough yet; or their teacher/the examiner is being particularly mean!

Unless the coefficients of a cubic polynomial function are judiciously selected it is unlikely that the function, or its derivative, will have rational roots. It is an interesting and challenging problem to find a means of *generating* non-trivial cubic polynomial functions with integer coefficients that have rational roots and also their derivative has rational roots. While there are general formulas involving radicals for solving cubic and quartic polynomial equations, they are not particularly convenient to use for by hand calculation, such as the quadratic formula, and no such formulas exist for polynomials of degree 5 or greater. So it would seem sensible to spend some time applying numerical techniques for solving equations involving polynomials of degree 3 or greater, pre-calculus this could be a converging interval approach, post-calculus Newton's method. Both methods can be conveniently implemented using CAS. Newton's method is based on selecting an initial point $(a, f(a))$ on the graph of f near to the intercept, finding the tangent to the graph at this point and obtaining the x intercept for this tangent. This is then used to determine the next point and the process repeated. The following is a simple example:

```
r[a_] := a - f[a]/f'[a]

N[NestList[r, -3, 10], 10]

{-3.000000000, -3.291666667, -3.253701342, -3.252983902, -3.252983648,
 -3.252983648, -3.252983648, -3.252983648, -3.252983648, -3.252983648, -3.252983648}
```

So the root is approximately -3.253.

Why Study (Cubic) Polynomial Functions?

There are several generally accepted reasons for including study of cubic polynomial functions in the senior secondary curriculum - pragmatic, conceptual/structural and application. These reasons are more broadly encompassed within the rationale for studying polynomial functions in the senior secondary curriculum. Polynomial functions are readily computable for rational values of the independent variable. If p is a polynomial function with rational coefficients, then computation of $p(x)$ for x rational only involves rational arithmetic operations for addition, subtraction and multiplication. In simple cases for low order polynomials where integer coefficients and values of the independent variable are used this can be done by mental computation, or by hand calculation.

Polynomial functions can be used in various application contexts. Examples of these can be obtained from the literature, and should be regularly incorporated throughout the study of these functions in the curriculum. These are not typically the kinds of applications that will be of immediate relevance to student's future everyday lives, but they are part of the repertoire of modelling functions that are used in a contemporary technologically advanced world. They include volume problems, modelling fuel flow, beam distortion under load in constructions, modeling distortion in lenses and medical imaging, Bezier curves in design, approximating flight descent and tracking paths. However, such applications (apart from the standard 'forming a box from an A4 sheet of paper' type) involve empirical data and related analysis, and are not likely to be amenable to the sort of by hand work that can be accessed from the sorts of selected examples referred to earlier. The use of polynomial functions for data fitting and modelling purposes has several limitations, for example, they are not good at modelling *asymptotic* behaviour.

For students proceeding to senior secondary study of a 'Methods' type course, polynomial functions are a 'new' type of function, but also a natural generalization from earlier work involving linear and quadratic functions, thus they play a conceptual connecting role. Polynomial functions are also algebraic functions that can be used as approximating functions (via truncated power series) for more complicated transcendental functions such as circular functions and exponential and logarithmic functions.

Some Observations

The study of cubic polynomial functions often takes place after some brief review of earlier work on linear functions and quadratic functions. While there is a connection between basic linear, quadratic and cubic functions in relation to measurement problems involving perimeter, area and volume respectively, this is only for positive values of the independent variable. The general and abstract treatment of linear, quadratic and cubic functions as polynomial functions provides an opportunity to highlight some common key ideas, however students can also come to see these functions as not particularly related – and instead view each with their own set of unrelated and idiosyncratic concepts and skills. For example, finding horizontal axis intercepts, as applicable, involves: for linear functions - simple solution of an equation of the form $ax + b = 0$; for quadratic functions - completing the square and solving for the independent variable *or* using the quadratic formula *or* factorizing; for (suitably selected) cubic functions the factor (little Bezout) theorem followed by polynomial division/re-expression and analysis of the quadratic factor.

For linear equations one can 'move things to the other side' in general for quadratic and cubic equations this is a 'no-no' (unless, for example, one is setting up a recursive approach to numerical root finding for a quadratic function). However it is also the case that a linear equation such as $3x + 7 = 0$ can be solved by factorization where

$$3x + 7 = 0 \Rightarrow 3(x + \tfrac{7}{3}) = 0 \Rightarrow x = \tfrac{-7}{3}.$$

The general consideration is that any polynomial function p of a single real variable can be expressed in the form $p(x) = (ax + b)q(x) + r$ and also as a product of linear and irreducible quadratic factors over *R* ('Methods' courses) and linear factors over C - the fundamental theorem of algebra ('Specialist' courses). The rational root theorem lets us know if there is a linear factor of the form $ax + b$ where a and b are integers, otherwise factorization isn't particularly much joy or utility in general. However knowledge of the general shape of graphs of polynomial functions of low degree as it relates to odd or even power of the highest power, and the sign of its coefficient is important, and the use of polynomials with rules expressed in form can be conveniently used for this purpose. For example, consider the graph of a cubic polynomial function with rule $h(x) = (x-2)(x-4)(x-6)$. This can be adapted to the form $h(x) = (x - m)(x - 4)(x - n)$ and CAS used with sliders for m and n so that m is varied from 2 to 4 to obtain while n is fixed at 6 to obtain the graph of $h(x) = (x-4)(x-4)(x-6) = (x-4)^2(x-6)$ and then n varied from 2 to 4 while m is fixed at 4 to obtain the graph of $h(x) = (x-4)^3$, and similarly for the graphs of some quartic polynomial functions, as illustrated in Figure 2 for $m = 4$ and $n = 7$:

```
Manipulate[Plot[(x - m) (x - 4) (x - n), {x, -2, 8}, PlotRange → {-20, 20}], {m, 0, 5, 0.5}, {n, 4, 8, 0.5}]
```

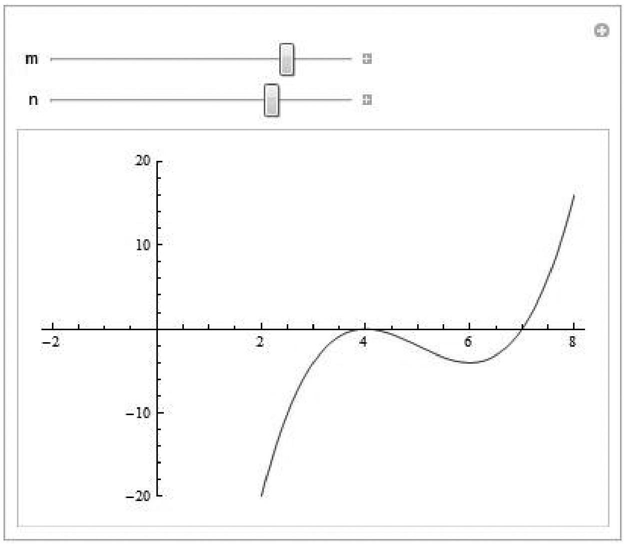

Figure 2. Graph with two sliders

Why No Simple Cubic Formula?

This is a reasonable question from the student point of view - the answer covers some interesting mathematics. All linear functions and their graphs are transformations of $y = x$ by a combination of dilation from the horizontal axis (possibly with a reflection in the horizontal axis), and a translation from the horizontal axis. This can be readily illustrated using sliders to transform the graph of $y = x$ and superimpose it on the graph of $y = ax + b$, for example $y = -2x + 7$, working through the algebra and interpreting the transformation at the same time. There is a simple linear formula for solving $ax + b = k$ as these transformations apply to a single occurrence of the variable and are readily invertible $ax + b = k \rightarrow ax = k - b \rightarrow x = \frac{1}{a}(k - b)$.

A similar argument applies for quadratic functions, all quadratic functions and their graphs are transformations of $y = x^2$ by a combination of dilation from the horizontal axis (possibly with a reflection in the horizontal axis), and a translation from the horizontal and/or vertical axes. Again this can be readily illustrated graphically, but is a bit more complicated, three sliders are required, and the algebraic machinery requires more development, including completion of the square, or equivalent. Once done this also leads to a formula obtained from inversion of the relevant transformations. Thus, from the completed square form $y = A(x+B)^2 + C$ where *A*, *B* and *C* are determined by the relevant transformations, one obtains for the roots of $y = 0$, $x = -B \pm \sqrt{\frac{-C}{A}}$.

Comparing the expansion of $y = A(x+B)^2 + C$ with $y = ax^2 + bx + c$ provides the relations $A = a$, $2AB = b$ and $C + AB^2 = c$. Using these relations to substitute for *A, B, C* results in the usual quadratic formula.

Likewise, other elementary functions are amenable to similar analysis, even if the inverse function that needs to be called upon is non-algebraic, for example with the sine function or the logarithm function. However this is not the case for cubic polynomial functions. The basic cubic $y = x^3$ only has a single stationary point, a point of inflection at the origin, and no amount of transformation of the kind considered above will change this into a cubic polynomial function with two distinct stationary points. So, no simple cubic formula and we have an interestingly distinct behavior. There is a cubic formula that can be obtained by other means - this is not straightforward, but CAS can be used to assist. There is the well-known and interesting historical narrative behind the algebraic problem of solving equations involving cubic functions – however it doesn't involve the factor theorem (see HREF1).

Building up by Addition

Addition of functions provides some insight into the nature of graphs of polynomial functions, and this technique could be used very simply in this context. The simplest polynomial is a constant function. A linear function is then a constant function with some x's added on; a quadratic function is a linear function with some x^2's added on; and so on. This can be illustrated by forming, for example, the graph of $q(x) = x^2 + 4x - 7$ from the graph of $f(x) = x^2$ and the graph of $g(x) = 4x - 7$. CAS technology can be used to produce lots of examples, from which students can, in complement with by hand sketching, generalize to note the location of key features, and follow through on related algebraic analysis.

This can then be extended to consider graphs of cubic polynomial function such as $p(x) = f(x) + g(x)$ where $f(x) = x^3$ and $g(x) = -4x^2 - x + 6$. What would have previously been tedious to do by hand becomes much more accessible when covered by an effective combination of technology assisted and by hand methods. Other decompositions are possible, for example, one could see what happens as 'each bit' is progressively 'added in'. Polynomial functions provide ready examples of the theorem that any function of a single real variable can be represented as the sum of an odd function and an even function, via the sum of odd its odd power terms and the sum of its even power terms.

Once calculus is involved other symmetries can be considered. For example it is well known that the graph of any quadratic function has reflection symmetry in the vertical line that passes through its vertex - while this is visually 'believed' how is it proven? The graph of any cubic function has half-turn rotation symmetry about its point of inflection (look at the graphs) – what would a proof of this look like? Finally, consider the graph of a cubic polynomial function with three distinct roots a , b and c . One can show that the tangent to the graph of the function at the midpoint of any two of these roots passes through the horizontal axis intercept that corresponds to the third root. This is illustrated in Figure 3 for $a = -4$, $b = 6$ and $c = 10$:

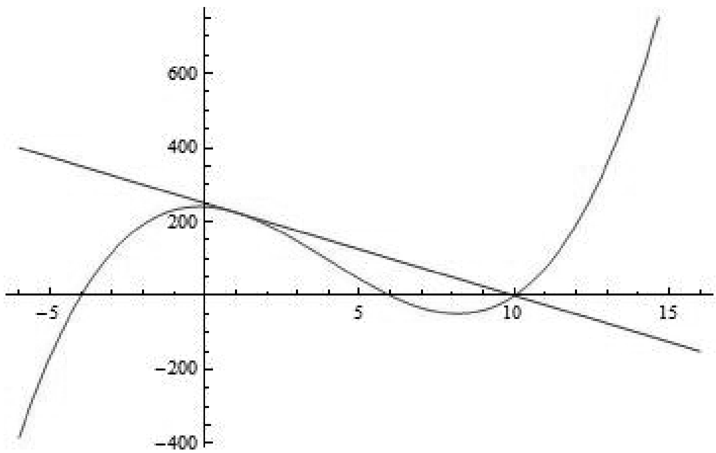

Figure 2. Graph with tangent through third intercept

This is a somewhat surprising result, empirically students and teachers asked to 'sketch' where the tangent at the midpoint 'goes' by and large do not have it passing through the intercept corresponding to the third root (try it with your class)

The result can be established in several ways, it is effectively a one-step application of Newton's method.

Some Concluding Remarks

Given the usual 'curriculum allocation' to the study of cubic polynomial functions and related graphs, algebra, calculus and applications it is worthwhile reflecting on whether 'established practice' provides a good range and balance of mathematical experiences for students. If generality is a key consideration for the concepts, skills and processes that we wish students to be familiar and proficient with, in terms of further study, work and application in STEM and other areas (but also more broadly as part of their intellectual heritage), then we need to carefully consider the extent to which we are making good use of the numerical, graphical and symbolic functionality of technology as to whether our curriculum intentions are being effectively realized.

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BIGGEST LOSER

Robert Money

Victoria University

This is an outline of the MAV's contribution to a cross-curricular unit of work that deals with issues associated with gambling, in particular the omnipresent sports betting and poker machines. Other members of the writing team are Ian Lowe and Donald Smith.

This mathematics unit covers the probability and statistics content of the Year 9/10 curriculum through activities that highlight the long term expectation of losses in various forms of gambling and the consequences of gamblers' misunderstanding of the independence of repeated trials.

The Context

Many of those teaching Year 9/10 mathematics courses spend two or three weeks, not necessarily around Melbourne Cup time, in covering the Year 9/10 probability and statistics content of the Australian curriculum. The mathematics part of the 'Biggest Loser' unit allows the outcomes to be met within the context of gambling. The context provides meaningful interpretations of key mathematical concepts, such as of expected values, independence of events, and the variability that underlies the maxim 'short term gain – long term pain'.

The closest the Australian Mathematics Curriculum gets to mentioning gambling is in the 'Personal and Social Responsibility' section of its 'General Capabilities'. Here it refers to mathematics education for 'making responsible decisions' and 'the development of students' personal and social capabilities by providing opportunities for initiative taking, decision making ---'. The ever increasing promotion of gambling cannot have escaped the attention of our students, adding gambling to the list of 'risky behaviours' that teachers need to deal with, notwithstanding any associated controversies (Smith, 2012a).

In 2010 one of the Productivity Commission Reports (2010, p. 9.20) found that

Little evidence has been collected about the effects of school-based gambling education programs on students' gambling behaviour. However, evaluations of similar programs in alcohol and vehicle safety have found that, while they can raise awareness, they tend to have no, or even adverse, behavioural impacts.

The report recommended that

Given the risk of adverse outcomes, governments should not extend or renew school-based gambling education programs without first assessing the impacts of existing programs. (p. 9.20)

Since then a great deal of research has been undertaken (Delfabbro, Lambos, King, & Puglies, 2009) and many gambling education programs have been developed, including one in Victoria which was distributed to schools in 2012 (Consumer Affairs Victoria, 2012). In consequence a key requirement in the development and trialing of the MAV unit was an evaluation of its implementation, in particular of its effect on the attitudes towards gambling of the participating students. The developers of the MAV unit were mindful to:

- exclude any element of the 'glamour and excitement' associated with the promotion of gambling; and
- exclude any activities likely to increase student accessibility to gambling sites.

The Mathematics Unit

The development of the 'Biggest Loser' cross curricular unit was partly funded by the Victorian Responsible Gambling Foundation, with MAV having responsibility for project management. The mathematics element is more developed than other parts of the unit. A summary of its draft form is as follows.

Lesson 1: Lucky Colours of Sunshine: Fair Game (Smith, 2012b)

Aims: To calculate probabilities for equally likely outcomes.
To explore short and long term outcomes in a fair game
To explore the concept of expected value as a sum of payouts times their probabilities.

The class decides on fair odds for a fair game. The use of a random number generator is highlighted as one strategy aimed to deal with misconceptions and betting strategies based on an inadequate understanding of independence of events.

Repeated play provides experience of random independent outcomes.

Results are analysed and then a computer simulation is used to generate long-term outcomes. Discussion leads to generation of a theoretical formula for expected value for the fair game and this is compared with the experimental results.

Lesson 2: Lucky Colours of Sunshine: Unfair Game

Aims: To explore the concept of expected loss on a $1 bet in an unfair game.
To explore short and long term outcomes in an unfair game

Students use simulation then analysis to obtain the expected loss from a $1 bet on an unfair game. A spreadsheet 'Going broke' will help students to see the rise and fall of their fortunes as the unfair payout is used and they can see how many games it takes to lose a 'pre-committed' amount of money.

Lesson 3: Sports Betting: Win, Lose or Draw in Soccer

Aim: To distinguish 'house odds' and real odds in calculating the expected loss on a $1 bet on the win/lose/draw outcomes for a game of soccer.

This activity uses the context of sports betting to distinguish 'bookies probabilities' - which add to more than 1 - from real probabilities. The relationships between payout, profit or loss, odds and probability are clarified. Students use commercial data to calculate 'house probabilities', real probabilities and the expected loss from a $1 bet. They contrast the expected player return of $1.00 for the fair dice game with 84 cents in most cases for the commercial data.

Lesson 4: Outcomes for Two Bets in Lucky Colours of Sunshine or Soccer

Aim: To use tree diagrams and calculations for combined events to represent and analyse probabilities for two bets on Lucky Colours or games of soccer. The activity uses these contexts to consider probabilities involving 'sampling with replacement' in two-step betting scenarios.

The results and concepts of the previous two lessons are used here to cover

- understanding of the probabilities of A or B, both A and B and neither A nor B (Venn diagrams, set notation, 'mutually exclusive' events)
- tree diagrams in relation to 'two step chance experiments with repetitions'

Lesson 5: Electronic Gaming Machines (Pokies)

Aim: To use a spreadsheet simulation to investigate expectation and variability in short term and long term betting on poker machines.

A spreadsheet is provided for pairs of students to simulate results for 100, 500 and 1000 one dollar bets on a poker machine set for a 12.5% loss to the punter. Students contrast the possibility of short-term gains (on 100 trials) with the near certainty of long-term losses (1000 trials). Box plots are used to illustrate variability and the unlikelihood of players winning in the long run.

Optional Review Lesson: Summarizing the Work So Far

Aim: To review the work done so far in the unit.

The lesson can be used to provide a poster showing worked examples of their own under a heading such as 'Different ways of losing money' or 'The 'house' always wins in the end'.

Lesson 6: A Quinella on the Stawell Gift Final

Aims: To calculate probabilities and bookies payouts for a quinella bet
To use simulation, tree diagrams and arrays to represent conditional probabilities involving equally likely outcomes

Students use dice to obtain small data sets that assist them in understanding what is required for a quinella. They are challenged to correct mistakes in an analysis that overlooks the 'without replacement' (conditional probability) aspect.

Lesson 7. A Quinella on the Melbourne Cup

Aims: To use a spreadsheet to develop payouts for quinellas for non-equally likely outcomes, providing a typical 30% margin for the bookie.
To distinguish between a 'payout' of $10 and the matching odds of 9 to 1
To see the difference between 'bookies probability' and real probability

The spreadsheet provided shows how payouts for winners (Melbourne Cup 2012 data) are used to calculate 'fair' payouts, based on real probabilities (which add to 1), for a quinella. Dividing each bookies' probability by their total gives the real payout.

Lesson 8: TAB

Aim: To better understand how the TAB betting system works.

A spreadsheet simulation pretends that 100 punters each have $1000 at the start of a race day. They all bet on each race with the TAB. The spreadsheet shows how the TAB takes its percentage first, and then the remainder (the prize pool) is shared among those who bet on the winner

Lesson 9: Putting it all together

Aim: A review of the important ideas covered in the unit.

Lesson 10: Gambling Systems Simulation – delayed assessment opportunity

Aim: To use a poker machine simulation to test misconceptions about independence and the fallacies in various betting strategies. This provides opportunity for students to reveal whether or not they still hold any misconceptions about probability and gambling.

Sample Lesson: Sports betting: Win, Lose or Draw

Aim: To distinguish 'house odds' and real odds in calculating the expected loss on a $1 bet on the win/lose/draw outcomes for a game of soccer.

Summary

The teacher provides data on soccer win/draw/lose payouts for $1 bets on soccer matches. Students then calculate 'house probabilities', real probabilities and the expected loss from a $1 bet. They contrast the expected loss of nil for the fair dice game with 16 cents loss in most cases for the data provided below.

Australian Curriculum/AusVELS

Probability AC level 9

225 List all outcomes for two-step chance experiments, both with and without replacement using tree diagrams or arrays. Assign probabilities to outcomes and determine probabilities for events

226 Calculate relative frequencies from given or collected data to estimate probabilities of events involving 'and' or 'or'

278 Collect data directly from secondary sources

Organize data on advertised payouts from a soccer betting site. For example,

Match	Home team win	Home team lose	Draw
Sunderland v Man Utd	6.00	4.00	1.57
Arsenal v Reading	1.25	6.00	11.00
Man City v Newcastle	1.30	5.50	10.00
Southampton v Chelsea	4.00	3.50	1.90
Swansea v Tottenham	3.10	3.50	2.25
West Ham v West Brom	2.30	3.25	3.20
Wigan v Norwich	2.00	3.50	3.25
Everton v Stoko	2.00	3.50	3.75
Aston Villa v Liverpool	4.75	3.60	1.25
Fulham v QPR	1.85	3.50	4.20

The recommendation that the teacher provide the data is to avoid 'grooming' of students towards actual gambling. What the site calls 'odds' are actually payouts. For example, a payout of $1.57 on a $1 bet makes a profit of only $0.57. There are many concepts to be understood. Most have been introduced previously, but there are new ideas.

- The difference between payouts and profits.
- The conversion of house profits to house odds.
- The conversion of odds to probabilities (a part to part ratio converted to a part to whole ratio)
- The realisation that the sum of the house probabilities is more than 1, and that this 'extra' is the expected house profit for each dollar bet.
- The conversion of the house probabilities to estimates of 'real probabilities'

The expected payout for one match is calculated as a class activity. This uses the real probabilities, not the house probabilities.

Arsenal versus Reading 31 March 2013	Home team Win	Home team Loss	Draw
Payouts, x	$1.25	$6.00	$11.00
House odds	4:1 on	5 : 1	10 : 1
House Probabilities, $1/x$	1/1.25= 0.8	1/6 = 0.1666	1/11 = 0.0909
Sum of house probabilities = 1.0575575---			
Real probabilities with a sum of 1 Divide by 1.0575575	0.75645	0.15759	0.08596

Calculation method

Expected return on a $1 bet:

E = $(0.25 x 0.75645 + 5.00 x 0.15759 + 10.00 x 0.08596) – $1.00

= $1.8366 - $1.00

= $0.84

In this real betting game, in the long term you expect to lose 16 cents in the dollar.

In the work session students use a prepared worksheet to complete similar calculations for other matches.

Some key questions for discussion during and after the work session are:

1. If the payout on the $1.00 bet is $1.25, then how much have you won?
2. Why do we subtract the $1.00 here?

How do you turn house probabilities to real probabilities?

What exactly does that expected value of 84 cents mean?

3. How do you get from house odds to house probabilities?

What values did you get for the sum of the 'house probabilities?

4. What is wrong with the house probabilities?

How should your real probabilities compare with the house probabilities?

How do your results compare with those for the Arsenal v Reading game?

5. What values did you get for the expected long term loss?
6. What do you need for calculating expected values?
7. How do you think betting agency decides what odds to offer on a soccer game?
8. How much money do you think is bet on each of these soccer matches, and what would be typical values of the amounts of money won by the 'house' and lost by the punters?
9. How much do they expect to win from punters who together wager one million \$1 bets?
10. What could go wrong for the betting agency?

Conclusion

Mathematics teachers have the skills to teach the probabilities relating to gambling, and the rising accessibility of gambling makes it more urgent that they should accept this responsibility. The mathematics element of this project is designed to address issues about gambling while at the same time covering the Year 9/10 content of the Australian mathematics curriculum. Ideally this can be done in collaboration with English and humanities teachers in a combined effort to increase students' understanding of the facts and the issues involved.

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Mathematics
of Planet Earth 2013

- Summaries

UNDERSTANDING THE EARTH IS A SUNDIAL

Tim Byrne

Casual Relief Teacher

The sundial is an ancient time keeping device. To be invented, the ancient mind needed to conceive the Earth itself as a sundial. The modern mind can possibly grasp the significance of this leap in imagination best when constructing a simple equatorial sundial based on first principles.

How pleasing it is to the modern mind to mimic the ancient insights into time keeping and be able to accurately calculate the hours of daylight with a handmade clock. Anno's (1985) book, *The Earth is a Sundial* makes it possible to firstly understand the theory behind sundials and then demonstrates how to construct one. Anno began his working life as a primary teacher and it shows in his expository style. He demonstrates how to make a horizontal and then in turn a vertical sundial from first principles. In my workshop I prefer to focus on the equatorial sundial. All sundials are derived from the equatorial sundial.

As Anno explains in his book, the Earth can be viewed as a sundial if one stands at either pole. Here during the summer months, the Sun remains above the horizon for 24 hours each day and as one rotation represents 360 degrees, each hour can be viewed simply as a 15 degree segment of a circle based on the Earth's axis. In other latitudes one can use the 15 degrees of the Sun's apparent movement but obviously with much less daylight to divide up. The mathematics is simple to this point. The next adjustment required for construction of an equatorial sundial requires an understanding of latitude. Latitude is usually viewed as parallel lines around the girth of the Earth's sphere. In fact these lines of latitude are calculated as angles rather than distance from the equator. By using the centre of the Earth as the origin of the angle, we can calculate that Melbourne is 38 degrees and the South Pole is 90 degrees south of the equator. For an equatorial sundial to be in tune with the turning of the Earth, it operates accurately at one latitude.

By aligning a gnomon to the Earth's axis means one need line up the sundial on a north-south line with the southern tip pointing directly at the south celestial pole. At the poles the sundial needs to have its gnomon pointing vertically, as the axis extends directly 90 degrees overhead. In Melbourne, this means ensuring the sundial is tilted at 38 degrees. At night, the south celestial pole is the imaginary point around which all the stars seem to rotate. The ancients used to call this 'the pole of the heavenly sphere'. In a long exposure photograph, the south celestial pole is defined by a multitude of concentric circles. In Victoria, the Southern Cross constitutes five of these concentric circles. However, in the northern states parts of the circles disappear below the horizon. When seeking direction at night, the Southern Cross becomes a means of locating the south celestial pole.

Recently I have used a CD (see Figure 1) to model the situation at the poles as described by Anno. A discarded CD with one writable white surface is best of all. The gnomon passes through the central hole to mimic the Earth's axis. One then only needs to tilt the CD and its gnomon to one's latitude for a successful working model. Of course one needs to find a way of securing a stick in the 15mm CD hole and setting it at the correct angle of one's latitude. My method for securing the dowel /gnomon is to use two rubber grommets on either side of the CD plate. Importantly, these grommets grip the dowel and allow some adjustment for length.

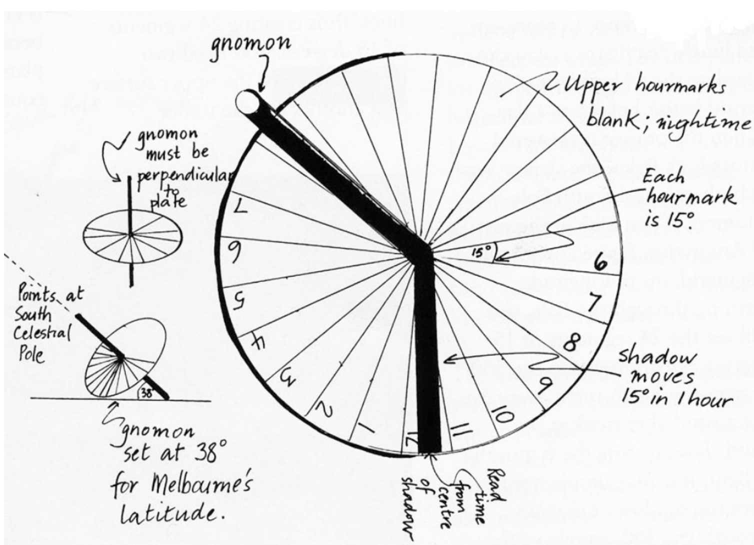

Figure 1. Equatorial sundial marked in 15 degree segments.

The invention of the sundial required two separate feats of the imagination. The breakthrough insight was realizing that the sky is an inverted bowl and a sundial needed to physically reflect that bowl. The ancients realized that if they could reproduce the track of the sun across the sky with a scaled model they had an accurate time keeping device. The second major leap of imagination concerned the gnomon's placement. The gnomon needs to be parallel with the Earth's axis or be raised to the angle of local latitude. The ancients adjusted the gnomon to point to the celestial pole, the still point in the heavenly sphere around which all the stars revolve.

Sundials enable one to grasp the big picture view of time. They invite thinking about how we calculate time through the triangular relationship between the Sun, the Earth and the sundial. The sundial shadow presages the night. Night follows day, day follows night *(lux et umbra vicissim)*; this is probably the most fundamental border we cross. Many sundial mottos capture the symbolism of the moving shadow to warn us. These mottos are a minor art form in themselves.

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PROMOTING MATHEMATICS LEARNING AND TEACHING THROUGH EMPLOYING FORENSIC SCIENCE (EAR PRINTING) AS A PBL DEVICE

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Problem-based learning (PBL) is argued through literature to be one of the most effective learning and teaching approaches across various disciplines; mathematics is no exception. This paper showcases PBL in mathematics education using forensic science settings. Forensic science scenarios (e.g., ear printing) can be employed as a PBL device to promote mathematics learning and teaching.

This paper argues that employing real and interesting problems – such as forensic problems – in the learning and teaching of mathematics creates an efficient and powerful pedagogical approach. Such an approach promotes students' mathematical understanding, critical thinking, and problem solving skills in an exciting and engaging learning environment.

Introduction

Problem solving is a fundamental part of constructing mathematical concepts, understandings, and competencies (Booker, Bond, Sparrow, & Swan, 2010). The growth of mathematical understandings in real-life problem solving contexts, which prompt engagement, impose challenges, and create motivation, is the hallmark of authentic and effective mathematical pedagogy (Bobis, Mulligan, & Lowrie, 2009). Hence, what would be more effective than promoting problem-solving through an engaging and interesting problem based learning (PBL) approach?

With the fascination of students with anything which relates to forensics or forensics shows such as the "CSI show" and its derivatives (e.g., CSI NY, Bones, etc)., mathematics and science educators are attempting to make use of such fascination in promoting students' understandings and capabilities in mathematics and science at upper primary and secondary levels (AAFS: Electronic). Such attempts are no longer adopted on an ad hoc manner. Departments of education in many countries are setting initiatives which integrate forensic science in mathematics and science teaching. For instance, The Department of Education in The Australian Government incorporated in its initiative "Scientists in Schools" a forensic science component in the school curriculum for upper primary-secondary levels which aimed to promote students' scientific understandings and capabilities (Howitt, Lewis, & Waugh, 2009).

This paper showcases PBL in mathematics education using forensic science settings. Forensic science (e.g., ear printing) can be employed as a PBL device to promote mathematics learning and teaching. The use of real, authentic, and interesting problems in the learning and teaching of mathematics – without reasonable doubt – is an efficient and powerful pedagogical approach. Such an approach promotes students' mathematical understanding, critical thinking, and problem solving skills in an exciting and engaging learning environment.

Problem-Based Learning

PBL is defined as "focused, experiential learning organised around the investigation and resolution of messy, real-world problems" (Torp & Sage, 1998, p. 14). Clarke et al. (1998, p. 5) define PBL as "authentic learning where students are driven to develop and test solutions to real problems". Delisle (1997, p. 1) describes PBL as a "discovering-learning process" which helps students internalize learning and gives them the chance to develop their own questions and investigative techniques. Maggi Savin-Baden (2003) argues that PBL is not only a teaching and learning strategy, but also a curricular approach in its own entity, context, and culture, where team learning and active learning practices are promoted.

Throughout the literature, many educators have argued the significance of learning through problem solving and the significant concepts and knowledge that a student may acquire through this method of learning. Barell (1995, p. 131) mentions that "Problematic situations are robust in that they contain within them significant concepts worth thinking about". John Abbott (1996) argues that the new competencies (e.g., abstraction, systems thinking, experimentation, and collaboration), which are essential for our ever-changing world go far beyond the old – but necessary – competencies (e.g., numeracy, literacy, calculation, and communication). These new competencies can be successfully acquired through the ability to conceptualise problems and solutions (Abbott, 1996).

Learning occurring in conventional teaching methods often includes listening, writing, observing and memorizing; whereas, learning taking place in PBL incorporates a much broader type of knowledge acquisition and application including active thinking, performing and experiential learning by trial and error (Barrows & Tamblyn, 1980). In the same context, Hmelo-Silver (2004) argues that psychological research and theory suggest that learning through a problem solving format facilitates learning not only of content, but also of thinking strategies. Over the last three decades, "the framework for understanding the psychological basis of learning has shifted gradually from a teacher-centred approach to a student-centred approach" drawing more attention and prominence to PBL (Sungur & Tekkaya, 2006, p. 307).

The Forensic PBL Device

This article will explore employing a particular forensic science problem to facilitate learning and teaching of Measurement. Ear prints will be used as a showcase for the efficiency of PBL, particularly when the problems used are authentic, interesting, and engaging.

Ear prints can assist criminal investigation in arresting suspects, particularly when such prints are one of the little evidence that can be developed from a crime scene. Usually, ear prints are found at a crime scene when the offender needs to listen by leaning against a window or a door to obtain information from within the premises. Such information may either be to confirm that nobody is present inside (e.g., burglary) or confirm that their victim is physically inside (e.g., assault).

The identification of an ear print on a window or a door helps in estimating the stature of the person who left this print (Lugt et al., 2005). Such estimation assists a criminal investigation in the elimination and/or inclusion of potential suspects. To be able to estimate a person's stature from an ear imprint s/he left on a surface, the following steps need to be followed:

- The height of the ear imprint from the floor is measured (denoted as H_1).
- The distance between the middle of the auditory canal and the top of the skull is estimated (denoted as H_2).
- The centimetres lost from the inclination when leaning towards a window or door to listen is estimated (denoted as H_3).
- The extra centimetres added by the heels of the shoes is estimated as well (denoted as H_4).
- The estimated height of the person (denoted as H_T) is then the result of adding up H_1, H_2, and H_3 and subtracting H_4: $H_T= (H_1+H_2+H_3) - H_4$. H_2, H_3, and H_4 vary from one individual to another. Age, stature, and gender contribute in such variation (Lugt et al., 2005). However, average values can be obtained in such measurements in order to minimise percentage error and provide a more reliable estimation of the stature.

The average distance between the middle of the auditory canal and the top of the skull is 13.4 cm (Hammer & Neubert, 1989; Hirschi, 1970). The average inclination when listening is 4 cm (Lugt et al., 2005). The average centimetres added by the heels of shoes are 2 cm for men's shoes and 4 cm for women's shoes. These average distances will not be provided to the students up until the end of the activity, for students will be required to conduct the measurements and compute the average values for each component of the stature formula.

Implementing the Forensic PBL Scenario

Students (pre-service teachers, College of Education, Victoria University) were challenged to solve the following problem:

Mrs Smith reported a burglary. All her jewellery was stolen. Sarah from the CSI Squad attended the scene: the thief broke and entered the Smiths' residence and left without leaving any trace except an ear print on the front door at a height x cm from the ground.

The police are interrogating three suspects with very little evidence: Suspect A (166 cm), Suspect B (177 cm), and Suspect C (184 cm). Sarah is required to assist the police in their investigation. Can you help Sarah?

Discussion – in groups – was initiated about how an identified ear imprint at the victim's front door at a height x cm from the ground can assist in estimating the stature of the individual who left it. These discussions led the students to identify the four elements (H1, H2, H3, and H4) contributing to estimating the stature: HT= (H1+H2+H3).

A printed photo of an ear imprint was attached to the white board and the students were required to measure the height of the imprint from the ground to identify H_1. The photo of the ear imprint was exactly left at a height of 163 cm.

The students were then – in groups – invited to estimate H_2, H_3, and H_4 by conducting measurements on themselves and on the members of their groups:

- estimating how many centimetres they will lose when leaning towards the wall,
- finding the average distance between the middle of the auditory canal and the top of the skull, and
- estimating the extra centimetres gained from their shoes

Extending the Activity

This activity can neatly fit – pedagogically and content wise – within the measurement activities planned for students in the upper primary- lower secondary levels (5-6 & 7-8 grades). The activity can be further extended to middle and upper secondary levels by:

- including computation of percentage error between the average values obtained from research (e.g., Hammer & Neubert, 1989 and Lugt et al., 2005) and those values obtained by the students,
- discussing the factors contributing to percentage error and variation in measurement,
- asking students to investigate any correlation (Pearson Correlation Coefficient and Coefficient of Determination) between the various dependent and independent variables in the activity such as the stature of the individual versus the inclination s/he makes as attempting to listen through a barrier (e.g., door or window).

Observations

When this activity was facilitated in a classroom setting (First Year pre-service teachers at Victoria University) as a demonstration of a PBL pedagogical approach, it was well received by all students. Students were engaged and excited. They posed their own questions and worked in groups to manage the challenges they faced and conduct the required measurements. Questions beyond the topic were articulated. These questions focused on percentage error, sources of error, and certainties and uncertainties in mathematics, science, and scientific investigation.

Conclusion

This article explores the realistic problems and exciting settings that forensic science brings about to PBL in mathematics education. The forensic flavour added to the activity motivated the students as it drew their enthusiasm and connected to one of their popular shows: CSI. However, the article does not – in the least bit – advocate for forensic science as seen from a "CSI Hollywoodian lens". On the contrary, this activity is designed in a manner which promotes students' appreciation to the certainties and uncertainties within science and mathematics (Samarji, 2013).

In addition to the broader goal of the activity in approaching a real sense of mathematics and science with all the certainties and uncertainties, the PBL activity focuses on developing mathematical thinking and understanding of measurement through a problem-based setting which prompts a range of generic skills including: problem solving, critical thinking, and collaborative team work. The aim of this paper is not limited to the employment of ear printing to facilitate the learning and teaching of measurement. The paper's aim goes beyond ear printing and forensics to advocate for PBL in mathematics education, where authentic, exciting, and engaging problem-based activities will leave a permanent print of knowledge, enthusiasm, and learning and teaching experience for both the students and the teacher.

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MEANINGFUL MEASUREMENT

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One of the most resistant questions in educational conversation ponders the reasons why students continue to struggle to retain information of a mathematical nature. One line of thinking of theorists suggests students struggle to link the highly atypical nomenclature and theory of mathematics to their increasingly more eclectic life experiences. This is especially evident amongst students who experience specific areas of mathematical deficiency, for example poor estimation skills. As tasks which have elements of intrigue, mystery and arriving at a personally constructed end product, mathematical units of work such as these should appeal greatly to boys. Many boys can become disengaged with mathematics at an early age and from then onward become 'maths-avoiders' because they fail to understand how mathematics applies to them in the real world and accordingly 'switch off' in class (Johnson & Gooliaff, 2013). This compounds their inability to understand how to solve/approach mathematical problem-solving scenarios. This unit on 'Meaningful Measurement and Scale' aims to demonstrate how mathematical rigour can be demystified and made more accessible for resistant learners and the mathematical challenged which in turn will create more flexible thinkers in mathematics. The key focus of the unit is the incorporation of mathematics in a context to which students can immediately relate: a housing measurement and house design activity. From a mathematical perspective it employs estimation, scale drawing competencies, paper models design and Google Sketch-Up. Students are also shown how to use mathematical computation to estimate potential costing predictions for the refurbishment of their house – carpet, paint and kitchen installation. The second activity, 'Temperature rising', emerged following news reports of young children and pets being left locked in cars on hot days by parents/owners who failed to understand issues related to surface area and heat loss.

The language of mathematics and the toolkit that it provides are often not grasped by the students who struggle to understand mathematical concepts and acquaint some relevance to the learning of these skills. Currently, students who undertake apprenticeships and traineeships as their chosen post Year 11/12 pathway are required to pass screening and selection tests, including the demonstrated competency of moderate to high levels of mathematical thinking. The contextual mathematics underpinning this unit of work goes some way to encourage students to embrace problem solving strategies, promoting alternative-solutions discovery (Meyer, 2010).

Crisp (2012) defines formative assessment as a tool that is designed primarily to improve learning, and summative assessment as that which exists primarily to judge learning. Often this is forgotten when assessing at the college level, as summative assessment becomes prioritised at the expense of the more valued formative assessment. Historically, mathematics employs maintenance rehearsal where students are asked to recall information that is repeated over and over through the completion of similar questions. Instead, the investigative approach of this 'Meaningful Measurement' unit seeks to activate elaborative rehearsal using Craik and Lockharts levels of processing, giving students a retrieval cue when they have to revisit topics (Eysenck & Keane, 2000).

Incorporating more meaningful investigative mathematical tasks into high school and college education will provide increased opportunity for students to gain confidence in their abilities and advance to more complex mathematical problem solving scenarios. Research by William et al. (2004) supports this and the longitudinal study reviewing assessment strategies and pedagogy demonstrates that student responsibility for learning and the choice of authentic assessment activities were reflected in improved student performance. The measurement unit aligns with the Australian Curriculum in both Number and Algebra, and Measurement and employs skill-building in measurement, Pythagoras and trigonometry, geometric reasoning, and strengthens number and place value to further develop the proficiencies (understanding, fluency, problem solving and reasoning). The unit concentrates on the development of size, shape, relative position, visualising two-dimensional and three dimensional objects in space. It highlights the importance of developing meaning behind measurement of quantities and choice of the appropriate units (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2013).

Lessons attached to the first stage of this unit are based around the students designing their house plans. It first uses a game where students have to match the house plan with the street view (to add complexity, the area of each plan can also be included). This promotes discussion about key elements of a plan and how houses should be represented when being placed on the market for sale. This learning could readily transfer to non-maths related subjects; the unit could be used in conjunction with an English class looking at advertising and the persuasive techniques advertisers use to highlight specific aspects of a house whilst excluding or carefully phrasing those areas of the house that are less desirable. This is in alignment with the teaching for understanding Harvard framework as stated by Dredge and Marsh (2012).

Following the preparatory study students then draft their own 'dream house'. Provision is made for students to first search the internet as a means of finding house design inspiration. A rough draft of their house, without measurements, is completed so all students have a structure for their house. Once completed, students are asked to add measurements, modifying their drafts as they develop a greater understanding of space. When incorporating measurements in their plan, students measure classrooms, desks, doorways and whiteboards as points of comparative reference. The garages and sports courts attached to the house designs proved the most difficult for students to arrive at measurement estimates, as there was little understanding of the difference between 25m, 50m and 100m. It seemed the larger the measurement the less comprehension students had of the relativity of area – the larger the area of space the more difficulty the students had with perception of space. Students measured staff car parks and the sports courts at the school for comparative spaces in their house design and objects to be filled in their house were configured using desks measured in the classroom. By using spatial imagery, which involves students creating a mental representation of "the spatial relations amongst objects, parts of objects, locations of objects in space, movements of objects and other complex spatial transformations" (Blajenkova et al. as cited in Moran et al., 2011), the visualisation of the house and rooms within the larger design became much more concrete. With this, the abstractness of their original design was removed and simplistic mathematical applications in their 'real world' could be positively transferred to the more theory based procedural processing component of this mathematical unit of work.

Once these competencies had been achieved, higher mathematical computation skills could be nurtured. The area and the perimeter of the house were calculated, with students asked to incorporate shapes other than rectangles into their houses. There were the more standard circles, and triangles where students used Pythagoras to find the hypotenuse, though several more irregular shapes such as hexagons and trapeziums which many students could not name, were involved. For the more irregular shapes, students investigated the formulas and attempted to link the uncommon shapes formulae back to the standard formula they had studied. The importance of shape and the students' lack of knowledge surrounding the name of shapes became particularly evident once the shape had been discussed. However, students talked to their peers in the class about the kinds of shapes they had used, engaging in active peer-tuition and student-led learning (Light, Micari & Pazos, 2010).

Plans were then drawn to scale by relating the task to work they had previously completed on ratios. Students became noticeably more eager to independently initiate a constant 'revise and polish' editing program of their drafts which served to strengthen further memory and procedural links to current and past mathematical topics further deepening their retention of the concepts and problem-solving strategies learned in mathematics to this stage of their education.

After the plan had been drawn to scale, students made their houses to scale out of paper and cardboard. This offered students more opportunity to master scale and also proved beneficial in visualising the plan. Students also gain a great sense of accomplishment when they can visualise mathematical theory actually resulting in some tangible product. Later on when students need to calculate paint usage, this model can prove very useful for those who struggle to visualise a way of seeing all the faces of their design. Obara (2009) further highlights the importance of using visual objects either as models or nets for students whose strength is not spatial learning. This approach also gives students time to reflect on what they have created so far and opens up more opportunities for discussion between student and teacher. It is also an important 'marketing tool' for the parents, visiting teachers/ students etcetera which debunks the idea that workplace mathematics rarely produces substance of a credible and sophisticated nature.

Following on from this, students then utilised software such as *Google Sketch-up*, the *Bunnings* 3D kitchen planner and used flooring and paint calculators and converters. This prompted positive unit conversion as some sites used millimetres rather than metres and discussion stemming from this led to further investigation of 2, 3 and 4 dimensions, the budget behind renovating or building a house, home loans and interest rates. The use of the available online resources was both practical and beneficial in highlighting the cost of building and renovating houses and offered students an important life skill as most will be active in the house market on leaving college. iPads, iPhones and the android market offer a range of apps which allow students to take measurements of their own house, providing the opportunity for them to make a scale plan. Renovation shows such as *The Block* could offer further extension to this unit where the students could 'compete' against the competitors in designing a room, considering the costs of the room and coming up with a plan of what renovation/refits they would make in their design.

The housing task can be followed by the 'Temperature's Rising' activity where students calculate the surface area of a 'child' doll and a pet left locked in a medium size car. Students are asked to look at the rate at which they can cool down. The measuring itself requires students to be precise and for them to come up with which shapes they could use to measure in order to arrive at some estimation for the doll; leaving room for the extension of leading students to developing nets of their 'child'. This again offers more opportunities for students to gain a wider awareness of shape, further developing their mathematical competency in problem solving and reasoning. On calculating the rate at which the child/pet cools down, students can then move into graphing and linear modelling units/investigations.

The unit gives the opportunity for feedback that is immediate and constant. As a formative task it is nonthreatening for students who are struggling. By immersing them in an activity that is contextually relevant, it encourages perseverance as students are keen to arrive at the final outcome. This multi-faceted learning approach aims to meet the learning and assessment needs of the students more effectively (Jenkins 2010) than do some other approaches. The units of work can be a useful tool by which targeted educational areas of disadvantage can be re-engaged, particularly for distance and rural education, indigenous students, or targeted programs to increase retention of students in post-secondary mathematics. This approach has been used in high schools and colleges with positive results and the idea of incorporating more formative assessment, although not a new idea, is constantly reviewed in teaching pedagogy articles with similar positive results.

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Housing Investigation - Student Instructions

- Design a dream house in which you would like to live
    - First, make a list of all the rooms you wish to include
    - Make a draft plan including all necessary measurements*
- Redraw the draft of this plan **to scale** (e.g.. 1cm represents 2m [1:200])
- Calculate the area and perimeter of all rooms.
- Draw your house in *Google Sketch up*
- Make your house out of paper, i.e., with the bird's eye view and add the walls. It all must be to scale.
- Using the Bunnings Kaboodle kitchen planner organise what your kitchen will look like and how much it will cost http://www.kaboodleplanner.com.au/UI/Pages/VPUI.htm
- Calculate how much it would cost to paint and carpet the whole house, use the Dulux Paint calculator to find out how much paint you will use. http://www.dulux.com.au/products/paint-calculator
- You can buy either: 500mL = $7 4L tin of paint =$77.59 10L=$147.50

*For a personal challenge and bonus marks include irregular shapes (circles, compound shapes)

Temperature Rising

Around the world, young children and pets are often left in cars while their parents or carers quickly leave them to do odd jobs unaware of the degree to which the child or pet registers the internal temperature increase. In December of 2012, SBS news reported that a baby died in the Victorian regional city of Bendigo after being left inside a car. Reports of animals being found in states of distress due to being locked in cars are also regular news occurrences.

a) Why could this be happening? Surely the internal temperature of the car is similar to the outside.

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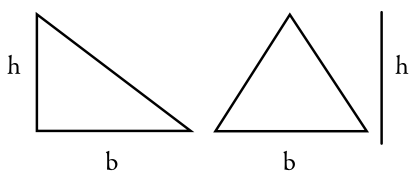
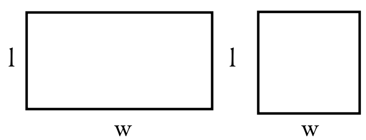
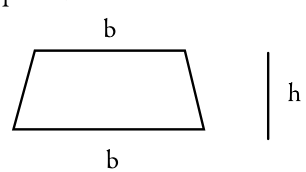
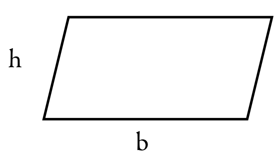
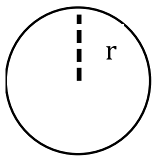
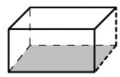
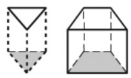
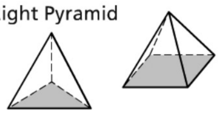
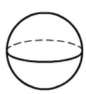
b) In small groups find the surface area of the 'child' and of one your group members. Do you think that a child is going to have a smaller or larger surface area to body mass ratio than an adult? Why/why not?

Part 1:

Calculate the surface area of the child doll and then calculate the surface area of one of your group members. Don't forget to show your workings out. Below are some formulas to help you complete the task.

Shape	Formulas for Area (A) and Circumference (C)
Triangle 	$A = ½ \times \text{base} \times \text{height}$
Square or rectangle 	$A = \text{length} \times \text{width}$
Trapezoid 	$A = ½\ (\text{sum of bases}) \times \text{height}$
Parallelogram 	$A= \text{base} \times \text{height}$
Circle 	$A = \pi \times r\ 2$

Rectangular Prism 	$V = lwh$ = length × width × height $SA = 2lw + 2hw + 2lh$ = 2(length × width) + 2(height × width) + 2(length × height)
General Prisms 	$V = Bh$ = area of base × height SA = sum of the areas of the faces
Right Circular Cylinder 	$V = Bh$ = area of base × height $SA = 2B + Ch$ = (2 × area of base) + (circumference × height)
Right Pyramid 	$V = \frac{1}{3}Bh = \frac{1}{3}$ × area of base × height $SA = B + \frac{1}{2}P\ell$ = area of base + ($\frac{1}{2}$ × perimeter of base × slant height)
Right Circular Cone 	$V = \frac{1}{3}Bh = \frac{1}{3}$ × area of base × height $SA = B + \frac{1}{2}C\ell$ = area of base + ($\frac{1}{2}$ × circumference × slant height)
Sphere 	$V = \frac{4}{3}\pi r^3 = \frac{4}{3} \times \pi \times$ cube of radius $SA = 4\pi r^2 = 4 \times \pi \times$ square of radius

Sourced from http://www.docstoc.com/docs/86958369/Formulas-for-Volume-_V-_-and-Surface-Area-_SA_-Figure-Wisconsin

Part 2:

Find the surface area to body mass ratio of the child doll as well as your group member

You will need to know the surface area and an estimate of the mass of the group member you have measured and the child doll. The greater the surface area to body mass ratio the more heat that is absorbed.

- The average body mass of a 3 month old, of similar size to your 'child', is 6.1kg.
- For your group member make an estimation of their weight using the chart.

Charts sourced from: http://www.alfitness.com.au/article/aID/120/cid/3/Find-out-your-ideal-weight-based-on-your-height

MENS CHART			
Height (centimeters)	Small frame (weight in kgs)	Medium frame (weight in kgs)	Large frame (weight in kgs)
158	58 - 61	59 - 64	63 - 68
160	59 - 62	60 - 65	64 - 69
163	60 - 63	61 - 66	64 - 71
165	61 - 64	62 - 67	65 - 73
168	62 - 64	63 - 69	66 - 74
170	63 - 66	64 - 70	68 - 76
173	64 - 67	66 - 71	69 - 78
175	64 - 69	67 - 73	70 - 80
178	65 - 70	69 - 74	72 - 82
180	66 - 71	70 - 75	73 - 84
183	68 - 73	71 - 77	74 - 85
185	69 - 74	73 - 79	76 - 87
188	70 - 76	74 - 81	78 - 89
191	72 - 78	76 - 83	80 - 92
193	74 - 80	78 - 85	82 - 94

WOMENS CHART			
Height (centimeters)	Small frame (weight in kgs)	Medium frame (weight in kgs)	Large frame (weight in kgs)
147	46 - 50	49 - 54	53 - 59
150	46 - 51	50 - 55	54 - 60
152	47 - 52	51 - 57	55 - 62
155	48 - 53	52 - 58	56 - 63
158	49 - 54	53 - 59	58 - 64
160	50 - 56	54 - 61	59 - 66
163	51 - 57	56 - 62	66 - 68
165	53 - 59	57 - 63	62 - 70
168	54 - 60	59 - 65	63 - 72
170	55 - 61	59 - 66	64 - 73
173	57 - 63	61 - 68	66 - 75
175	59 - 64	63 - 69	68 - 77
178	59 - 65	64 - 70	68 - 78
180	61 - 67	65 - 72	70 - 79
183	62 - 68	67 - 73	71 - 81

Compare the cm per kilogram for the 'child' and your group member. Is there any difference?

Check your answer using the Du Bois' formula

$S=71.84W^{0.425} L^{0.725}$

S= surface area in cm^2

W = body weight in kg

L = body length in cm

Part 3:

What about dogs? Can I leave my Jack Russell in the car?

Find the surface area to mass ratio for a small dog. Is it different from a child?

(25-35cm) (6-8kg)

Part 4:

Ibrahim Almanjahie from the University of Western Australia (2008) found that on a typical summer's day, the temperature in a parked car can rise to more than 20oC higher than the temperatures outside the car.

Table 1. *Rise in temperature*

Time in car	**10 minutes**	**20 minutes**	**30 minutes**	**60 minutes**
Temperature rise windows shut	10.6 °C	16.1 °C	18.3 °C	24 °C
Temperature rise windows open	9.4 °C	11.7 °C	13.3 °C	21.1 °C

a) Draw a graph representing this information

b) Draw two lines of best fit

c) Using your lines find out the temperature rise after 40 minutes
   i) With the windows open?
   ii) With the windows closed?

d) How long does it take to rise 20 °C
   i) With the windows open?
   ii) With the windows closed?

e) What else do you think might affect the rate taken for the car to heat up?

Table 2. *Temperatures in Hobart*

Month	Hobart min average temperature (°C)	Hobart maxaverage temperature (°C)	Hobart highest recorded temperature (°C)
January	11	22	40.8
February	12	22	40.1
March	11	21	37.3
April	9	18	30.6
May	6	15	25.7
June	5	13	20.6
July	4	12	22.1
August	5	13	24.5
September	6	15	31
October	7	17	34.6
November	9	19	36.8
December	11	20	40.6

If the internal temperature of the car reaches above 40 °C negative health effects will occur for the child or pet.

Using Table 1 and the Maximum average temperature in Table 2, calculate which months and how quickly the temperature will take to rise above 40 °C. Record how fast it takes to reach the over 40 °C assuming the windows are shut.

Month	How long does it take to get over 40 °C?
January	
February	
March	
April	
May	
June	
July	
August	
September	
October	
November	
December	

Part 5:

The average 12 month old child is 10.3kg, the average adult male is 85kg and average female is 70kg

A 12 month old child loses 1 to 2 mL/kg per hour through perspiration

Adults lose 10 to 20mL/kg per hour

How long does it take a child to lose as much sweat as an adult? Figure it out mathematically. You can use any method.

Part 6:

Does the colour make a difference?

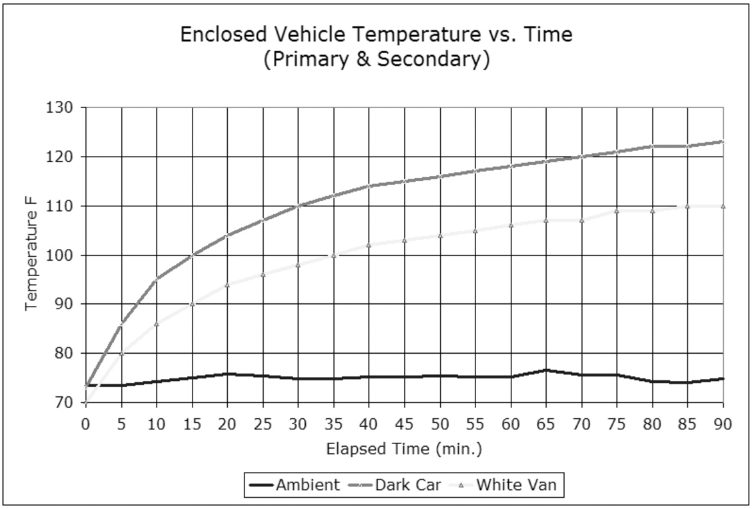


Figure 1 sourced from http://books.google.com.au/books?id=PCU0RwDI6c4C&pg=PA40&lpg=PA40&dq=body+size+and+heat+loss+people&source=bl&ots=syZL-iK7Xi&sig=ouwOvIl-BcuiN2zbQWCylmpgbZg&hl=en&sa=X&ei=V0NzUbioI-SRiQft4oDwBg&ved=0CEoQ6AEwAzgK#v=onepage&q=body%20size%20and%20heat%20loss%20people&f=false

a) Using Figure 1 above, what was the difference in temperature between the dark car and the white at

   i) 30 minutes?

   ii) 60 minutes?

   iii) 90 minutes?

b) Using the Fahrenheit to Celcius online converter http://www.metric-conversions.org/temperature/fahrenheit-to-celsius.htm calculate the difference in temperature in degrees Celcius

   i) 30 minutes

   ii) 60 minutes

   iii) 90 minutes

c) Does the size of the car make a difference? If I have a larger car, is it safe to assume that I can risk leaving my child or pet in there for longer?

   i) Design a way that you could test this
   ii) If I leave a bowl of water in with them does it make a difference?

HOW DO WE ENCOURAGE STUDENTS TO GET THE RIGHT BALANCE BETWEEN CAS AND BY-HAND IN VCE MATHEMATICS?

Sue Garner

Ballarat Grammar

The new Casio ClassPad II is a wonderful tool to use in the VCE Maths classroom but we still come across students who use it to draw simple graphs when they would be better to do it quickly by-hand and, in contrast, other students who are afraid to explore the ClassPad in the problems that seem overwhelming for them.

This session will outline some approaches that are useful in the classroom when preparing students for VCE examinations using CAS technology. The paper will use examples by demonstrating some features of the new colour Casio ClassPad II.

Introduction

2002 was the first year that Year 12 students in Victoria sat their Mathematical Methods examinations using a Computer Algebra System (CAS) as part of their external assessment. The new VCE subject Mathematical Methods (CAS) paralleled the existing Mathematical Methods subject from 2002 to 2006. Approved CAS technology was permitted in examinations. Mathematical Methods ran as a separate subject from 2002 to 2009 and is no longer available. The 2006-2015 current VCE Mathematics Study Design has CAS technology assumed for school based assessment and includes technology active and technology free external examinations. CAS technology is now assumed in all three of Mathematical Methods (CAS), Specialist Mathematics and Further Mathematics Units 1 to 4.

The CAS Classroom

In the CAS classroom we see improved display of data and graphs, a new personal authority and changed classroom social dynamics and a change in the didactic contract. CAS produces a different classroom with a new mathematics authority. The technology empowers students to become a source of information to share about mathematics and about technology. The 'explosion of methods', produced by the calculator, gives students more to contribute and more to consider.

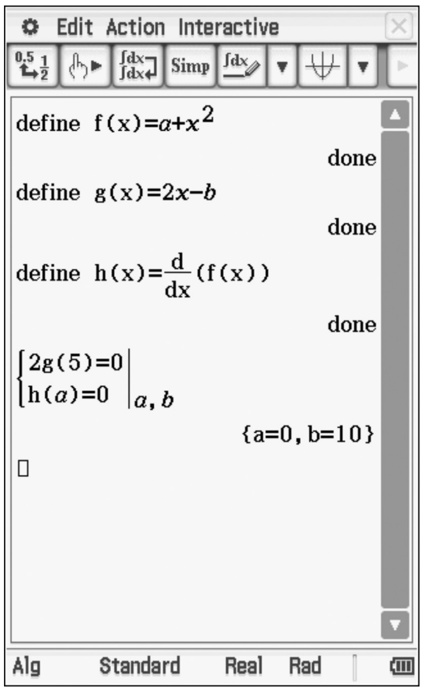


Figure 1. Use defining to solve difficult problems.

Students can engage in solutions of real world problems, scaffolded by CAS.

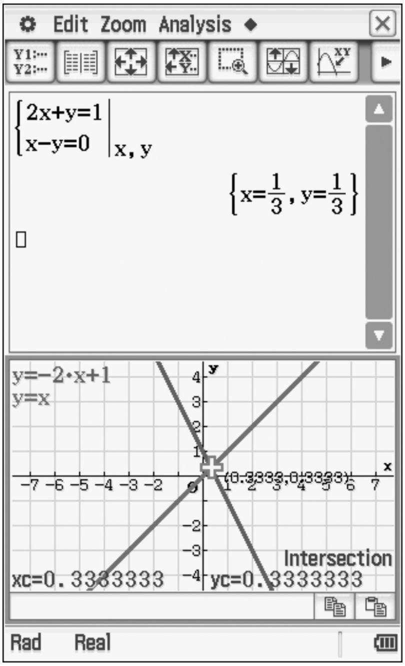


Figure 2. Use simultaneous equations before learning solution methods.

It is possible to take students on a 'magic carpet' ride to get a bird's eye view of a new topic, before looking at the details. Students can get a 'macro' view of mathematics by encapsulating a multi-step process as a single command.

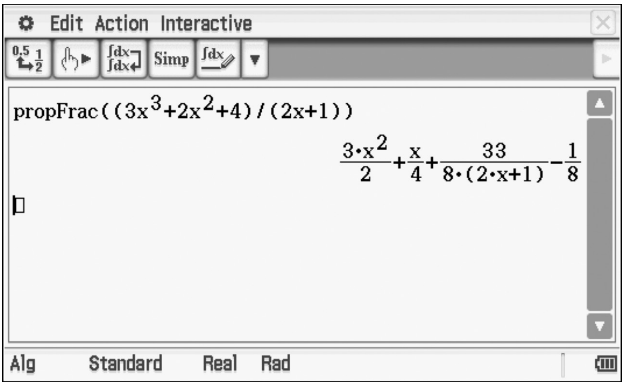


Figure 3. Use propFrac for polynomial long division/rotate screen.

By-hand Algebra

How do you decide when algebra should be done by-hand or symbolically with CAS? Reflections from 10 years of experience tell us that there is no significant difference in by-hand skills while students are using CAS. Some students use CAS to greater effect than others and there are factors that help or hinder. Garner and Pierce (2005) reported four groups of students: the *enabled*, the *stayers*, the *resenters* and the *flyers*. After ten years it seems that the clearest groups are now the resenters, where cognitive conflict is present, and the flyers, who are still finding things out for themselves at home and showing the class the next day.

What About the Technology?

The technology over the ten year period has become better and more user friendly, where previous technology issues that were unresolved are now resolved. Students and teachers are still game to try different things and teachers are learning alongside the students.

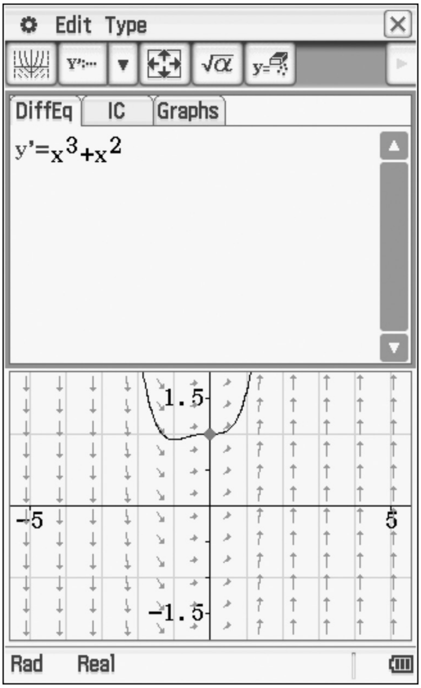


Figure 4. Use slope fields.

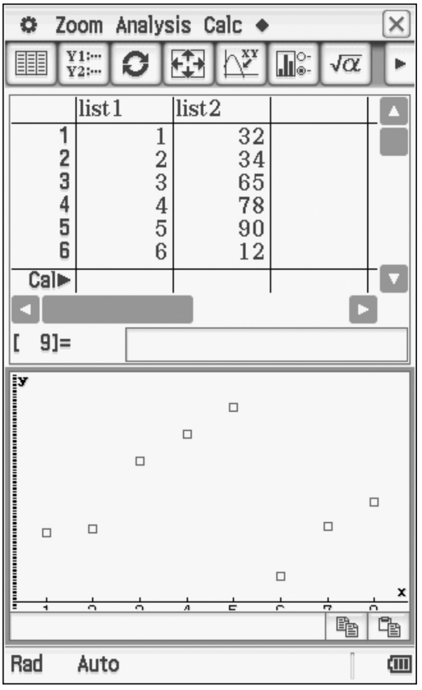


Figure 5. Use Statistics.

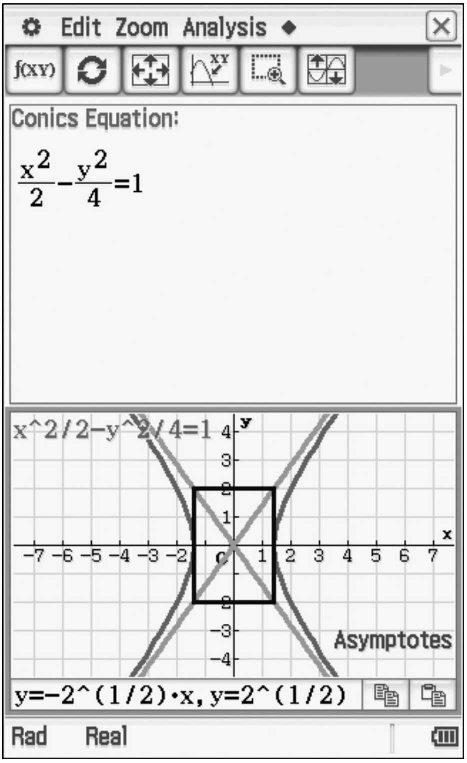


Figure 6. Use Conics.

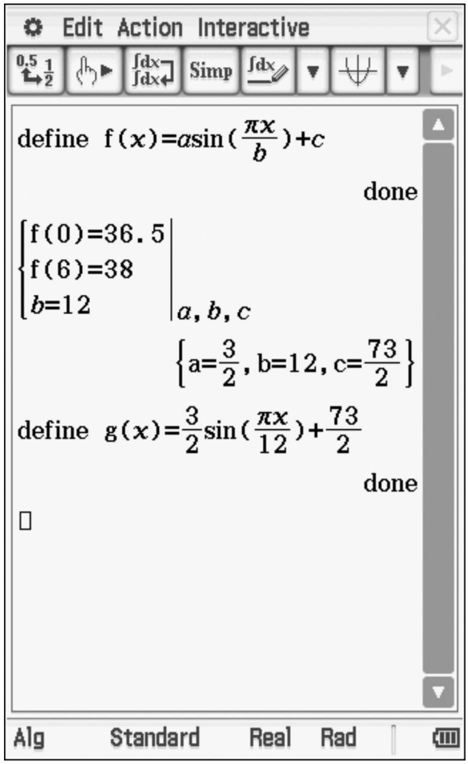


Figure 7. Use define and simultaneous equations where $f(x) = a\sin\left(\frac{\pi x}{b}\right) + c$

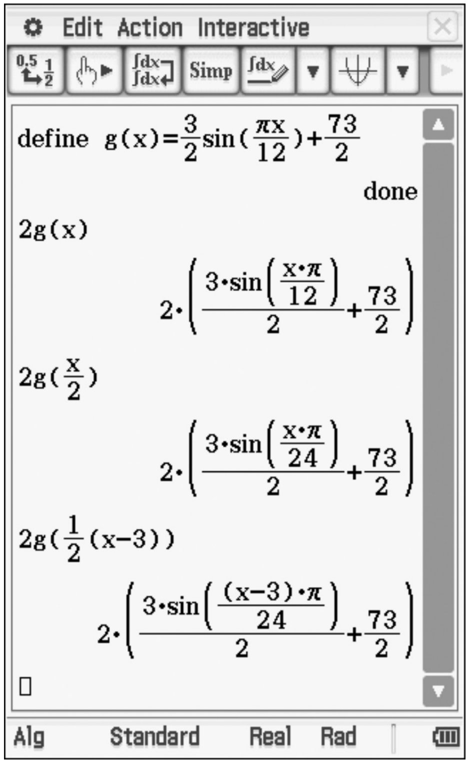


Figure 8. Use transformations.

Explore an Unknown Function

Consider the function $y = e^{-x}\cos(x)$. What do we do first? We can consider algebra patterns, sketch graphs, analyse the algebra and analyse the graphs. Students need to consider which comes first, and which is helpful.

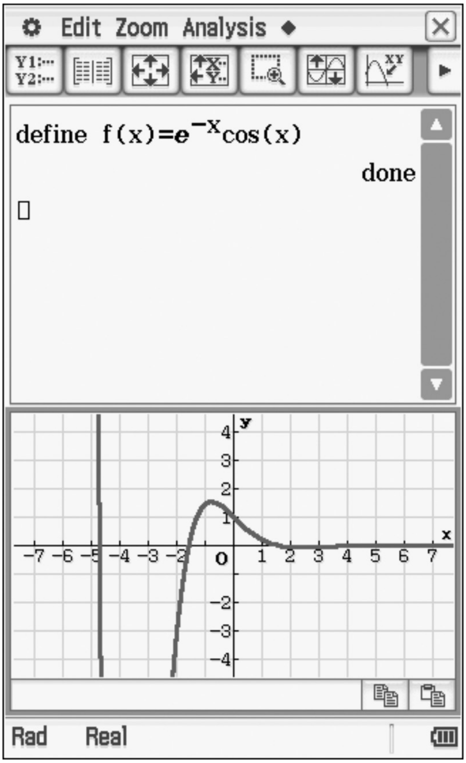


Figure 9.

How many students just draw a graph without considering algebraic patterns and why is this still happening? They can consider the zeros, asymptotic behaviour. What does an exponential graph do? What does a trig graph do?

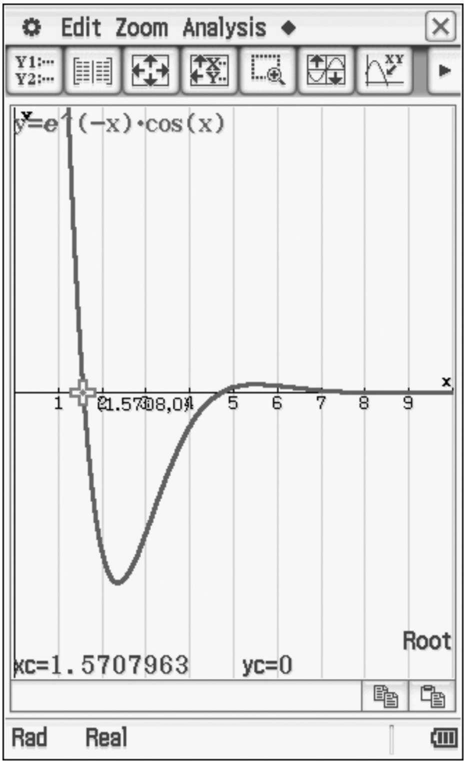


Figure 10.

Does it matter that a student doesn't know what an envelope curve is?

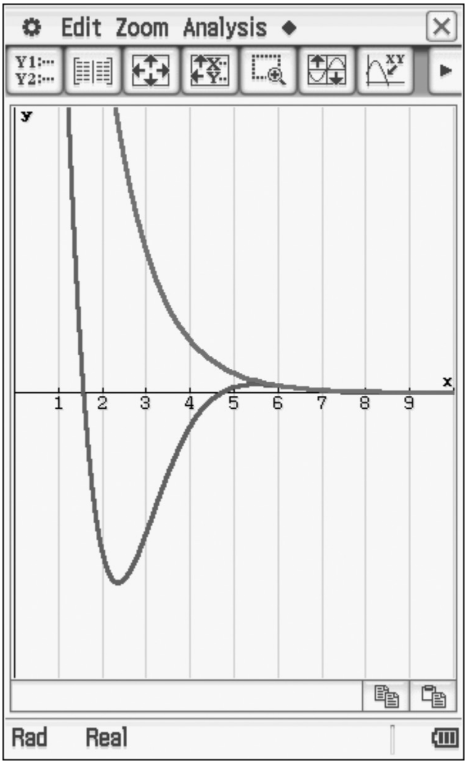


Figure 11.

Conclusion

CAS technology enables students to explore techniques and functions previously unheard of and as teachers we need to equally encourage the exploratory use of the technology. Too often CAS is used just to check answers and to draw graphs, but this is a tiny aspect of its availability and usefulness.

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WILL THE XMOOC'S TSUNAMI ENGULF - SECONDARY MATHEMATICS EDUCATION?

Brenton R Groves

Independent Researcher

The inline URL references have direct hyperlinks to their web pages. Download this summary from http://tiny.cc/MAV2013Summary or request a copy by email to make the hyperlinks interactive.

MOOCs were invisible to the Australian academic community until The Australian newspaper Higher Education Supplement published five articles on the 20th of February, 2013. Since then there have been articles almost every week and MOOCs have swept through tertiary education like an angry tsunami.

The acronym 'MOOC' was coined by David Cormier at a Canadian educational conference in 2008. His 4-minute video 'What is a MOOC?' <http://www.youtube.com/watch?v=eW3gMGqcZQc#> is a good starting point for newcomers to the field.

MOOCs have into two separate pedagogical directions: content-based xMOOCs and connectivist theory cMOOCs. This presentation defines the motivation behind MOOCs and lists xMOOC courses available for a teacher's professional development. A separate literature search for the pedagogical basis of cMOOCs can be downloaded from http://tiny.cc/CLSearch

What is an xMOOC?

xMOOCs are simply old-fashioned conventional university distance-education courses supercharged by the Internet Age. At present they are a personal development path for teachers but in the near future they will have an accreditation process to a MEd degree backed by Harvard or MIT and recognized by your local teaching authorities.

As made evident in this presentation, in some respects little has changed between the teacher and the student since J J Clark published The Correspondence School--Its Relation to Technical Education and Some of Its Results <http://www.jstor.org/stable/1633383> in Science Magazine in 1906. The International Correspondence School claimed an enrollment of 902,906 students from 1898 to 1906. Clark stated:

> Our system of education is based on an idea that is almost directly opposite the views held by the regular schools and colleges. .... The colleges demand that a student shall have certain educational qualifications to enter it, and that all students study for approximately the same length of time, and when they have finished their courses they are supposed to be qualified to enter any one of a number of branches in some particular profession.
>
> We, on the contrary, are aiming to make our courses fit the particular needs of the student who takes them. ... Such a student does not wish or desire to be forced to study anything that is not strictly necessary for him to learn in order to fill the position he is aiming at. .... We have absolutely no way of compelling a student to study. We cannot threaten him with suspension or expulsion.

What is different between ICS in 1906 and in 2013 is the delivery machinery. The Internet is universal, fast and cheap. Class size no longer matters. Web 2.0 allows the individual to access the world's knowledge. Most importantly, social media and BYOD (Bring Your Own Device) forms a virtual person-to-person classroom that is as effective as the real thing.

Further In-Depth Reading

Martin Davies, Can Universities Survive the Digital Revolution? Quadrant Online, Volume LVI Number 12, December 2012:

http://www.quadrant.org.au/magazine/issue/2012/12/can-universities-survive-the-digital-revolution

Yuan, L., & Stephen Powell, S. (2013), MOOCs and Open Education: Implications for Higher Education, White Paper 2013:WP01, JISC CETIS.
http://publications.cetis.ac.uk/wp-content/uploads/2013/03/MOOCs-and-Open-Education.pdf

xMOOC Courses for Professional Development

Many places offer MOOCs, and many more will. Coursera, Udacity and edX are the leading providers.

Table 1. *Hyperlinks to MOOC providers and course catalogs.*

MOOC Provider	Home page hyperlink	Course Catalog
Coursera	https://www.coursera.org/	https://www.coursera.org/courses
Udacity	https://www.udacity.com/	https://www.udacity.com/courses
edX	https://www.edx.org/	https://www.edx.org/course-list/allschools/allsubjects/allcourses

Further In-Depth Reading II

Martin Weller, Slide Show, 5 reasons to do a MOOC, slides 11-15. 5 reasons NOT to do a MOOC, slides 16-20.
http://www.slideshare.net/mweller/moocs-march-2013

The New York Times, MOOCs, The Big Three, in November 2, 2012 comparing Coursera, Udacity and edX. *http://www.nytimes.com/2012/11/04/education/edlife/the-big-three-mooc-providers.html*

The Best MOOC Provider: A Review of Coursera, Udacity and edX:
http://www.skilledup.com/blog/the-best-mooc-provider-a-review-of-coursera-udacity-and-edx/

Coursera Announces Professional Development Courses to Facilitate Lifelong Learning for Teachers:
http://blog.coursera.org/post/49331574337/coursera-announces-professional-development-courses-to

SCHOOL-BASED ASSESSMENT - INSIGHTS FROM QUEENSLAND SENIOR MATHEMATICS

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With the implementation of the Australian Curriculum for senior secondary mathematics pending, it is timely to reflect on the experiences and practices of our state and territory senior mathematics curriculums to ensure lessons learned from experience are not ignored. School-based assessment has been an integral part of the Queensland Education system for just over four decades. Strengths and shadows of school-based assessment should be considered to ensure the approach to implementation of the new Australian Curriculum is effective and efficient for all.

Australian Curriculum – The Future

With the content and achievement standards of the Senior Mathematics Australian Curriculum finalised and approved as a common base for development of senior secondary mathematics courses in December 2012, it is now the responsibility of the separate state and territory curriculum, assessment and certification authorities to determine and organize how the content and achievement standards will be integrated into senior mathematics courses around Australia. The Australian Curriculum, Assessment and Reporting Authority (ACARA) is working with state and territory bodies in 2013 to determine the process and timeline for implementation in classrooms throughout the states and territories in Australia. (HREF1)

Without specific intention of the timeframe for implementation in each state and territory, now is certainly a time that can be used by teachers in schools for considering, engaging, exploring, reflecting and planning for implementation while further reflecting and refining practices under current curriculum.

Queensland Senior Mathematics - Overview

In Queensland, the Queensland Studies Authority (QSA) develops, maintains, supports and oversees the range of syllabuses offered in Queensland schools. Schools have flexibility to design syllabus-based work programs to best engage their cohorts and reflect their resources. (Queensland Studies Authority, 2010)

Authority subjects contribute to the calculation of Overall Positions (OP) used for entry to university courses and Authority-registered subjects that are not used in the calculation of the OP. In the senior years, there are three Authority Mathematics Courses – Mathematics A, Mathematics B and Mathematics C; and Prevocational Mathematics is an Authority-registered mathematics course. (Queensland Studies Authority, 2010).

Queensland implements a standards-based approach to school-based assessment that is measured across five achievement levels from very high through to very low. In the senior years, student achievement is based wholly on school-based assessment items. Schools develop assessment programs and items to enable judgments about achievement standards to be made and assessment packages and judgments for Authority subjects are externally moderated each year by QSA, through state and district review panels (Queensland Studies Authority, 2010).

Levels of achievement are measured for each assessment item against a matrix that describes the standards for each dimension – knowledge and procedures, modeling and problem solving and communication and justification. The exit level of achievement for each student is determined by reference to individual assessment packages and their overall match to the standard descriptors within and between the assessment dimensions. (Queensland Studies Authority, 2010)

For each of the three authority-registered mathematics courses, school-based assessment in most schools includes two exams and one research or extended modeling and problem solving task each semester over the two year course. The process for marking of assessment items varies between schools but many schools use a cross-marking process to ensure consistency of decisions and feedback to students.

Comparisons - Australian Curriculum and Queensland

Without delving into analysis of direct alignment between the content four Australian Curriculum courses of General Mathematics, Mathematical Methods, Specialist Mathematics and Essential Mathematics and the four Queensland mathematics courses Pre-vocational, Mathematics A, Mathematics B and Mathematics C, the terminology of aims and assessment share many similarities.

While the Australian Curriculum prescribes two assessment criteria of concepts and techniques and reasoning and communication and the Queensland Mathematics Syllabuses stipulate the three criteria of knowledge and procedures, modeling and problem solving and communication and justification, much of the terminology in the descriptors is similar. As expected, descriptors require demonstration of knowledge, understanding of concepts and selecting and applying techniques in routine and non-routine problems. Australian Curriculum terminology uses "in a variety of contexts" (HREF2) and Queensland Syllabuses specify "life-related and abstract situations" (QSA, 2008). Use of digital technologies is common to both.

In relation to the communication of mathematics, there are also many commonalities. Where the Australian Curriculum refers to representing information in "numerical, graphical and symbolic form", "explains the reasonableness of the results and solutions", presenting arguments that are "succinct and reasoned" and "identifies and explains the validity and limitations of models" (HREF2), the existing Queensland mathematics courses incorporate "use of mathematical terminology, symbols and conventions", justification of the reasonableness of results", reasoning that is "coherent, concise and logical" and "evaluation of the validity of mathematical arguments in the context of problems" including the "strengths and limitations of models, both given and developed" (QSA, 2008).

Insights and Experience of School-Based Assessment

The benefits of school-based assessment can be seen to be allowing teachers to devise assessment items that enable students to demonstrate the full extent of what they have learned rather that teachers teaching to curriculum content that they predict will be on an externally devised assessment. Developing an assessment program that incorporates a variety of assessment techniques suits a diverse range of learning needs and learners. It also places teachers at the centre for learning and assessment, enabling them to create appropriate assessment items and deliver specific feedback for improvement and learning to the students in their care. (Queensland Studies Authority, 2010).

In his paper presented at the 32nd IAEA in Singapore, Maxwell (2006, p. 3) proposed that the benefits of school-based assessment include "attention to a greater range of important learning outcomes, opportunity for contextualised and authentic assessment, integration of formative feedback for improvement, and generation of an achievement profile over time rather than on a single occasion". He further recognised that these benefits correspond to anticipated demands on people and economies in the future.

The challenges of developing, administering and grading school-based assessment include issues such as equity, validity, reliability, authenticity, balance, comparability and consistency of decisions. While QSA provides guidance on the issues of equity, balance and authenticity, the responsibility of implementation falls on schools. The monitoring process is also governed by QSA but responsibility for staffing district and state panels also falls on schools and practicing teachers. While it can be appreciated that teacher release for involvement in district and state panels and other professional development opportunities increase teacher expertise in assessment in schools, school-based assessment does increase pressure for teachers both in terms of time and workload (McCollow, 2006).

In implementing a new Australian Curriculum, there will be increased workload in schools in developing new assessment items to reflect the changes in the curriculum. An increase in system support and available professional development is desirable to ensure all teachers in all schools have opportunity to develop proficiency in developing assessment items that provide opportunity for students to demonstrate their capabilities and thus achieve to their potential.

Conclusion

School-based assessment has been an integral component of the Queensland Education system since the early 1970s. It offers benefits for authentic learning, integration of feedback into the learning assessment loop and development of an achievement profile over time. It also places teachers as professionals at the centre of learning and assessment. In implementing school-based assessment, responsibility for issues such as equity, validity, balance and authenticity fall on schools. External monitoring processes address issues of reliability and comparability. In moving toward the implementation of the new Australian Curriculum in senior secondary mathematics, there will be increased workload developing assessment programs and items to reflect changes in the curriculum. System support and professional development is desirable to ensure all teachers in all schools have the opportunity to develop their proficiencies in developing and grading school-based assessment.

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PLANNING AND ASSESSING VCE MATHEMATICS

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We have collaboratively developed an organized approach to firstly summarizing the key knowledge and skills that are required to be assessed for each of three outcomes in the VCE Mathematics subjects we teach. This information is then used in spreadsheet form as the basis for our initial planning of assessment tasks and then finally to mark and provide feedback to students in rubric form. We teach both Unit 1 & 2 General Mathematics and Unit 3 & 4 Further Mathematics.

So What Should We Be Assessing?

This was the first question that we endeavored to address. To do this we looked to the current Victorian Certificate of Education Study Design for Mathematics and produced a spreadsheet that summarized the key knowledge and key skills for each of the topics/ modules that are part of the curriculum that we teach. Initially it was our intention to use this document to simply record manually what each of our students had demonstrated either through completing the set exercises from the textbook or through the other assessments in the form of tests and SACs. We then went through the process of allocating our already existing questions in these assessments to the key skills and knowledge in our spreadsheet. At this stage our motivation was that we wanted to be able to look at this record and be able to say to an individual student what they had demonstrated satisfactorily and what they had not. We figured this would give us a way of judging then whether students had demonstrated enough to be able to pass each outcome. Plus we would have lots of concrete evidence to back our judgments up if ever they were called into question.

How Do We Go About Assessing to Cover the Key Skills and Knowledge For Each Topic/Module?

The next stage of the process was to be able to incorporate this information into a rubric that we could use to mark the various School Assessed Coursework (SAC) for each of the topics/modules that needed to be assessed. Initially we followed the process of allocating our already existing SAC questions to the key skills and knowledge in our spreadsheet. What we began to notice was that some of the questions in the SAC focused repeatedly on the same skills and knowledge, while other skills and knowledge were completely unrepresented in the SAC. The next stage was then to start with the spreadsheet of key skills and knowledge and find the questions that would give students the opportunity to demonstrate each of these. This way we could then ensure that we had questions of the right type and value to ensure that we provided an assessment task that would be balanced and broad enough to cover all three outcomes for each topic/module.

Marking Assessments and Giving Feedback on Each SAC in Terms of Outcomes and Key Skills and Knowledge:

We used Microsoft Excel to produce a spreadsheet for each SAC, this spreadsheet has a page for each student that can be printed onto an A4 page once marking and moderation is complete. It also has a summary page for the class so a breakdown of marks for each outcome and total marks and grade can be easily identified and used in the reporting process.

BRINGING MATHEMATICS & SCIENCE TOGETHER

Rhonda Lyons

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1st Dimension	2nd Dimension	3rd Dimension
Number, counting. Cardinal ordinal. Measurement, Foundational. Length, Mass. Addition, Subtraction. Multiplication (linear) division. Fractions decimals, percentage. Singular quantities. Positive, negative number. Pattern & Order. Sequence and repetition Points on a number line, open number line. Number systems. Scales, Kelvin Scale.	2D area, shape, space. Geometry. Perimeter. Multiplication, (array) division. Graphing, mapping, Location, Symmetry. Data handling. Algebraic equations, Spreadsheet, Base systems. Money.	3D shape, design, space. Volume, capacity. Chance & Data. Probability. 3D Pie charts. Longitude, latitude. Structure, Architecture. Design & drawing. Algebraic functions. Robotics. Work, mechanics, motors, Engineering.

4th Dimension	5th Dimension	3rd Dimension
Time. Analysis. Physics, time and motion. Physical movement. Time & distance. Global navigation. Travel, speed. Economics, time & money. Economics linked to travel. Sustainability. Environment. Formulas & Functions. Generating & testing conjectures. Structure, set, logic.	Cyber space. Digital technology. Computer programming. IT, ICT. Logo, programming. Modern technology. Creative design. Digital design & spatial organisation. Calculus tools. Dimensions and applications. Engineering.	Hyper and Nano (extra large and microscopic) Splitting the atom. Medical science. Neuron science. Astrophysics. Space exploration. Astronomy.

The Big Picture is the scope and sequence chart of the Dimensions of Mathematics and Science Education.

Begin with Counting

v Addition. ----------------------------Subtraction v

v < (Inverse) > v

v Multiplication -----------------------Division v

2 ways to access division:

1. repeated operation of subtraction,
2. inverse operation of multiplication.

How many teachers use the simplest form of illustrating division which is repeated subtraction?

Small Scale Situation – The Teaching and Assessment Cycle

Numeracy Interview, P-2

- The teacher manipulates the materials and the pupil looks and answers.
- This is a visual style of assessment which assesses cognitive strategies and conceptual understandings of the student, but the pupil does not even write anything.
- Do we need the child to show the learning? How does this fit with constructivist learning theory?
- Year 3 assessment is online, On Demand testing, and then NAPLAN standardised assessment.
- This is a pictorial representation of the question or problem where the child selects an answer, 1 of 4 (A, B, C, D).
- NAPLAN: visual representation of number problems, with pictures. Select answer, one of four, colour the dot. This is assessing conceptual understanding of mathematics skills.
- Is the result consistent when the child has to manipulate the materials? Most classroom programs use numeracy a program which is far more manipulative. Is this necessary? Best practice is probably a healthy mix of the multiple styles of teaching and learning. Abstract, concrete, illustrated, and e-learning.
- Does the pictorial representation transfer to real application? Can the child show this?
- Can they write and create a problem and then answer it? Is this part of classroom activity?
- Money & Time, relevant to child but how important? Other considerations? Child's priorities.
- It is difficult to have authentic problem situations which are not constructed by the adult, or important to adult criteria. Are real number situations and applications relevant to the child?
- A skills based learning program around number is comprehensive because it does not have to have a context. It is knowledge that is broadly applicable to many applied situations.
- For the child, the learning program should have personal application, community application (may mean school setting) and / or (real) world application. This drives open ended learning, provides diversity for student engagement and challenges student to go further where possible and interested (extension).

Teaching and Learning Mathematics

- Teachers need assistance to read and interpret school level data and student level data.
- We should have a numeracy coach or leader available to each school, capable of assisting teachers with this task; also to moderate with teachers to draw conclusions from student assessment data.
- Plan a curriculum program to focus on skills development.
- Mostly number focus because that strengthens applied learning strategies, BUT applied mathematics should be a part of every classroom practice, weekly.
- Children will transfer knowledge as they develop conceptual understandings.
- E-learning is diverse and engaging.
    - If we are too prescriptive teachers do not have the opportunity to try, evaluate and grow as teachers. Action research at the basic level is vitally important to individual teacher development. Sharing understandings and knowledge with peers is vitally important also.
    - Beginning teachers should have a peer for team teaching in the first two years of practice for core learning in literacy and numeracy.
    - Learning is as much a part of teaching as teaching, intertwined, for years. Document and note developments in own understanding (teacher) as part of yearly review. Set goals and evaluate annually.
    - Developing core teaching techniques and practices. Validate with peers in mediated group work and interact with another classroom at regular intervals.
    - Requires some trial and error.
    - With coach or leader, teachers have to opt in – the desire to learn and improve is necessary for success.

Further readings

Constructivist Learning Theory & Cognitive Development.
Rhonda Lyons
Master of Education (Mathematics)
Deakin University

UTILISING AUSTRALIAN BUREAU OF STATISTICS (ABS) RESOURCES TO DEEPEN UNDERSTANDING OF DATA REPRESENTATION

Vivienne McQuade

Australian Bureau of Statistics (ABS)

The implementation of the Australian Curriculum (AC) gives impetus to deeply examining what teachers need to understand about data representation when designing teaching and learning programs. This paper examines the developing ideas and strategies in data representation at the Australian Bureau of Statistics (ABS).

Which Data Representation to Use When?

The value of presenting data in a range of forms so audiences are engaged to turn data into information and explore its meaning is widely acknowledged. A very important skill to possess, and a focus in many classrooms, is knowing which graph to use when. Technological advances also drive us to discern and explore which visualisation to use when and understand the purpose of data visualisation when we are the audience.

Effective use of the right data representation, or visualisation, is achieved when we understand how and why we want to present data. Some important questions to regularly ask in our classrooms are:

- What is the purpose of this infographic or visualisation and do you think it has been achieved?
- Would a different representation achieve different things?
- How important is this form of representation in the statistical investigation process?

- Which forms of data representation do you like the most?
- Which form of data representation is the most difficult for you to read?
- Which forms of data representation are the easiest to evaluate?
- Does the representation support you to analyse data and see patterns, trends or relationships between variables?

Incredible Data Representation

A key area of difference of how statistics was taught (and learned) in the past, and now, is the incredible power and excitement of technology. Whether the technology is a graphic calculator, an Excel program, a GPS, web Apps, gifs or a myriad of other tools, it is much easier to learn about statistics and acquire data knowledge because of this power and accessibility.

Data representation can involve data visualisation and infographics. This involves the visual presentation of data to communicate the stories contained in the dataset. Data visualisation can communicate complex information in a way that is easier to interpret by turning data into visually engaging images and narratives.

Data can be communicated visually through:

- Static visualisation - the use of tables, graphs, charts and infographics which provides a visual snapshot of the data. The user can view the data, but not interact with it.
- Dynamic visualisation - the use of animations which emphasise key information and show prescribed movements in the data. The user can follow and/or explore limited aspects of the data.
- Interactive visualisation - data users are able to change the graphics so as to view different variables. This provides opportunities for users to become active data explorers with the freedom to customise what they see, look deeper into specific areas of the data, or use motion to track patterns over time and space.

Data visualisation not only communicates data in a digestible format; it can also be used as a tool for data analysis. Patterns, trends, relationships between variables, and the distribution of the dataset can be more apparent in visualised form than when presented as numbers in a table. Data visualisation is beneficial when you need an overview of a dataset rather than the specific values contained in the dataset. The visualisation will highlight the stories in the data, or focus on selected data from within the dataset for a specific purpose.

ABS Use of Data Visualisation

The following presents some examples of how the ABS has harnessed the different types of data visualisation to present data in an engaging format for its users.

Static Visualisation

- Tables, graphs and charts are present throughout the ABS webpages in catalogue releases and products.
- CensusAtSchool has many resources which build skills in what graph to use when, including using Excel to create summary tables, charts and graphs.
- CensusAtSchool data is visualised in a suite of infographics, presenting national summary and time series data.

Dynamic Visualisation

- A population pyramid is a graph showing the age and sex distribution of a population. It is a useful tool in revealing information about a population's history and future possibilities. The ABS animated population pyramid shows the change of population distribution over time.
- Census population change is visualised by state and territory and for capital cities, demonstrating change from 2006 to 2011.
- The ABS Stats App allows you to choose summary data for key economic, population and Census data you require.
- On various ABS 'Topics@ a Glance' pages, you can explore data by state and territory through animated maps e.g., Crime Victimisation in my state, education and training statistics.

Interactive Visualisation

- The 'Run That Town' App is a game where students make decisions and the results of their decisions are immediately presented through animated popularity scores, bank balance, 'clout' and mock media reports.
- The Education Services' box plot tool allows students to input their own data sets and compare data at a glance, watching and interpreting changes in box plots.
- The ABS' 'Spotlight' on Census is an animated infographic into which people are invited to input their own data and 'shine some light, and see what kind of a story Census data can tell you about you'.

Conclusion

2013 is the International Year of Statistics. It is worth noting that in 1977 John Tukey said, "The greatest value of a picture is when it forces us to notice what we never expected to see" (Tukey, cited in Cotgreave, 2012). While technology has changed dramatically since 1977, the challenge to notice the unexpected has not reduced. Producing, explaining, applying and interpreting a range of data representations in our classrooms will enhance students' understandings of the power of statistics in our day to day lives as we notice the unexpected, form decisions and make choices that will have far reaching impact on all of us.

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CREATIVE APPROCHES TO SKILLS AND DRILLS

Jennifer Palisse

Mater Christi College

An enthusastic teacher once asked me for some ideas on parabolas for her advanced Year 10 class. I showed her a Phet simulation I had used with my physics class for projectile motion, available at http://phet.colorado.edu/en/simulation/projectile-motion (Fig. 1). It involves shooting objects out of a canon aiming for either a traditional bullseye, or a statue of Michelangelo. We explored how changing the initial speed or the angle of the canon affected the shape of the parabola. We posed questions such as "What family of parabolas would always hit the target?" or "What family of parabolas have the same height?" After bouncing ideas off each other for some time, she finally stopped and said "I'd love to do this, but I just don't have the time".

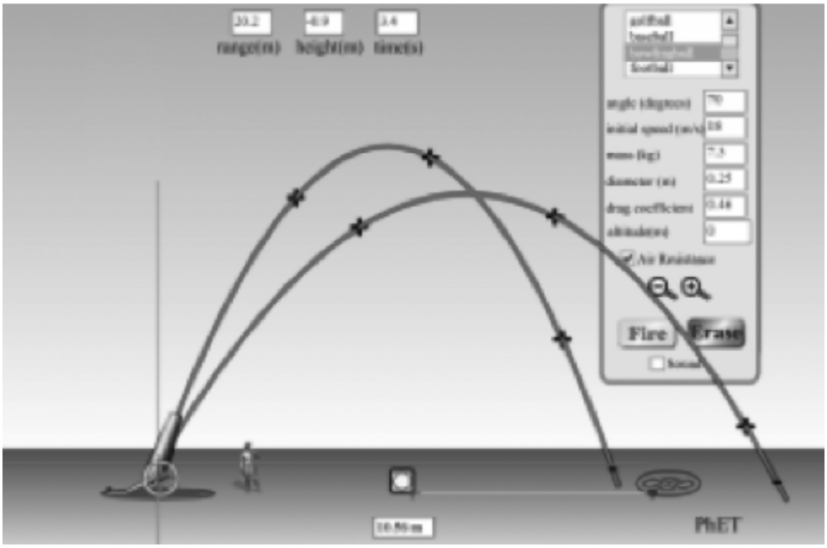


Figure 1.

Through conversations with other teachers, it would seem that many see value in student-centered learning but worry it takes too much time, the general reaction being something along the lines of 'you want me to get through teaching all of these skills and include lots of problem solving?' I agree that if student-centered learning is viewed as the addition of problem-solving to an already busy curriculum, it would take far too long to complete. Rather student-centered learning should be seen as situations where "students solve problems not to apply mathematics but to learn new mathematics" (Van de Walle & Lovin, 2006, p. 11). Both the learning and practice occur by solving poblems. It's a two-for-one deal: problem solving and skills practice all in one! This is where the timesaving comes from. It's about recognising that you can choose problem-solving tasks to include skills practice, without then having to also include excercises from the textbook.

What follows is an investigatory task on the Koch Snowflake I have used with a Year 7 class. While it may not be the best example of student-centered learning, as it does not require students to develop new concepts, it does highlight how an investigatory task can be used in lieu of the textbook, hence allowing plenty of time to incorporate both problem-solving and skills practice. Tasks such as these can offer a nice starting point in shifting towards a student-centered classroom.

The snowflake also has value in its ability to make links across several topics, something which is often lacking in textbooks (Vincent & Stacey, 2008). It incorporates fractions, perimeter, area of triangles, finding patterns, plotting coordinates, analysing data, limits, and depending on your level, special triangles and summation notation. This task was used with a poorly achieving group of Year 7 students and took just under a week to complete. The basic aim of the task was to provide students with plenty of practice on finding the fraction of an amount as well as practice finding the area of a triangle.

Drawing the Snowflake

First, have students create their own snowflake by following these steps:

Step 1

Draw a large equilateral triangle using triangular grid paper. Use a base length of any power of 3 (we used a base length of 81 to avoid difficult fractions).

Step 2

Divide each straight line into three equal parts and rub the middle part out. (Fig. 2)
Replace the middle part with a new equilateral triangle.
Repeat!

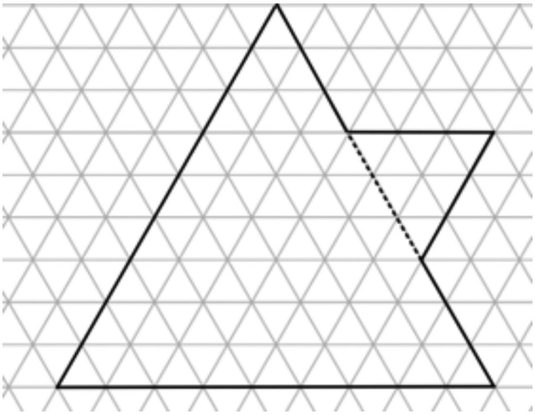


Figure 2: Beginings of stage 2

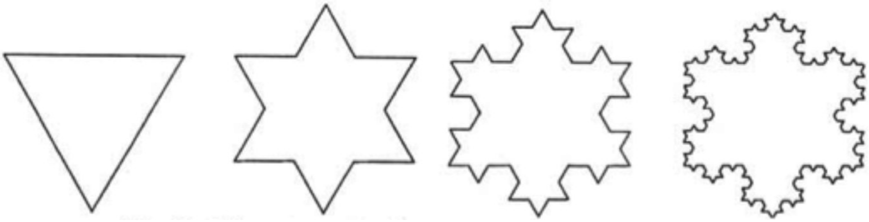
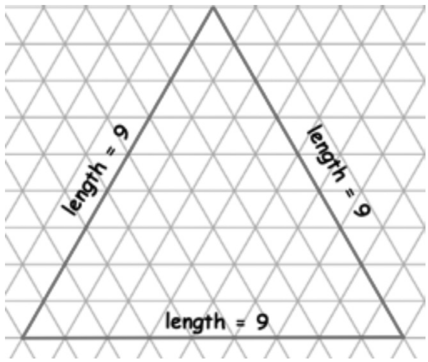


Figure 3. Stages 1 to 4 of the Koch Snowflake

Figure 3 shows the first 4 stages of the snowflake. See if the students can get to stage 5. Note that fractal designs can often be very challenging to get started, and it is suggested that you complete the first two or three iterations as a whole class. Make sure you've practiced one on your own before attempting it with your class and bring lots of extra grid paper for students to start again.

Brace yourself for some frustration as you let the students try a stage on their own. I found the mantra "chop into three and make the middle one pointy" useful. Some students will pick it up quickly and want to try and squeeze stage 6 and 7 at the microscopic level, while others will need a lot guidance simply getting to stage 3.



Finding the Perimeter

Have students calculate the perimeter for each stage. For our class, we measured the length of the line as the number of triangles (9 in the diagram above). Be careful that year 7s measure the   of the line, not the number of triangles along a row as many were tempted to say the length was 17. A starting length of 81 can be useful for a poorly achieving group as the perimeter will remain a whole number until stage 6. A starting length of 9 units can be useful for a medium group as they can practice the first few stages using whole numbers before tackling fractional lengths.

In our class, students were asked to share any strategies they used when finding the perimeter at stage 2. Care was taken over ordering their explanations to the class in such a way as to build on the ideas of the pervious method. The first group of students showed how they had physically counted up the number of triangles along each line. As this was a fairly low achieving class, several students had used this method and when we compared numbers between those who had also counted up the length, we quickly found that this method was inaccurate, with large variation in answers (remember, we started with a triangle length of 81 so our perimeter at stage 2 had a length of 324). These students all agreed that this method was inaccurate, cumbersome and took too long.

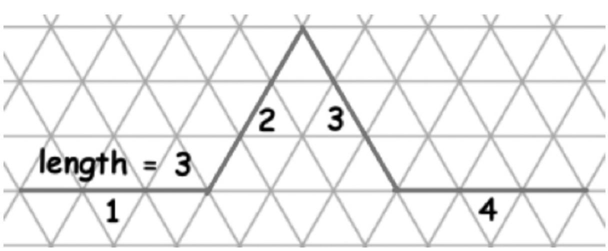


Figure 4. Breaking the Koch Snowflake into separate segments

A second group was asked to share their method. They first began by counting the length of one section: length 3. They then counted that there were 12 sections in total around the snowflake (four of the 12 sections are shown in Fig. 4). Noticing that each section had the same length, they then calculated $12 \times 3 = 36$. Those who had used the first strategy agreed that this method was far quicker and less likely to result in counting errors. But remember, I had chosen this task so that students would practise finding a fraction of an amount and neither strategy had mentioned anything about fractions yet!

The final group that was called on to present showed how they didn't need to count any lengths. They worked out what the length of one segment would be by finding one third of 9 to be 3. They then calculated the length of one side of the snowflake as $3 \times 4 = 12$ and finally trippled their answer to include all three sides: $12 \times 3 = 36$. Thankfully, someone had finally mentioned fractions. In order to encourage the whole class to use fractions, the students were asked to predict what the length of a single segment would be for the next stage without counting first. They were allowed to then check their answers by looking at their picture. This was enough of a push for all students to utilise fractions instead of counting.

Students were then left to calculate the perimeter for stages 1 – 4 and were asked to predict what the perimeter would be for stage 5. Again they could then check using their picture. Students were then challenged to describe a rule for finding the perimeter at any given stage.

The rule for the perimeter at stage *n,* with a triangle of original length *l,* ends up being $P_n = \frac{l \times 4^n}{3^{n-1}}$. While this may look complicated but the students were encournaged to write a separate rule in words for the numerator of the fraction and a separate rule for the denominator. Generally students did not see any recognisable patterns in the perimeter if they kept a table of values with just the final answer for their perimeter. Instead they were encouraged to include their working out in a table. This prompted them to leave their answers using 3s and 4s. For example, rather than recording stage 2 as 36, it was recorded as $3 \times 3 \times 4$. In this form students quickly recognised that the pattern had something to do with the number of 3s and 4s. Some naturally switched to index notation when comfortable.

Finding the area

Without a knowledge of surds and pythagoras, Year 7 students will be unable to calculate the exact area of the triangle. In our previous lessons, we had discovered how to calculate the area of a triangle using $\frac{1}{2}\text{base} \times \text{height}$ but had not practised any calculations. Calculating the area of the triangle was possibly the most difficult part of the task, aside from drawing the snowflake. Students had a lot of trouble determining which measurement to take for the height and many were unconfortable with the notion that the height wasn't actually one of the sides of the triangle. We decided to sacrifice my own snowflake and cut out two of the triangles to rearrange them into squares. From this, we were able to determine which measurement we needed for the height.

It is worth asking if the students would have found finding the area of the triangle difficult if they had been given a set of text book questions. A typical exercise might show a diagram of a triangle with only two lengths provided and it isn't so difficult to take the only two numbers given and substitute those numbers into the formula $A = \frac{1}{2}b \times h$. The decision of "what information do I need" has been removed and has instead been turning into a substitution quesiton (which could also be useful as a lead into algebra). Note that a formula for the area requires the limit of a series which is usually too difficult for Year 7s.

Analysis of data

As an addition to this task at a later date I would ask students to plot a graph for both the area and perimeter at each stage. This would require the ability to work out an appropriate scale for the axis by first examining the range of their data, but would also inform me of the students who can/cannot plot coordinates. There would also be some practice in converting fractions to decimals as students would be asked to express their perimter calculations as decimals. The general shape of the graphs are given in Figure 5, for a triangle with starting length 1.

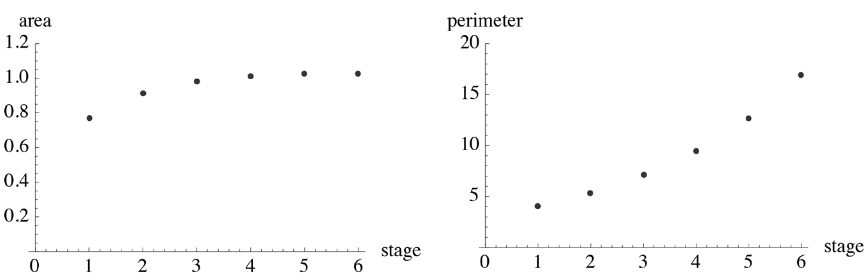


Figure 5. The area and perimeter of the Koch Snowflake for stages 1-6 with triangle of starting length 1.

These graphs can lead to a rich classroom discussion on limits, and the more bizarre idea that an infinite perimeter encloses a finite area. Students can be asked to draw the horizontal asymptote for the graph of the area and discuss its meaning. A nice real-life application of this is the antena: a device where the more wire you use, the stronger the signal, yet you want it to take up the least amount of space possible (Fig. 6).

Figure 6. A fractal antena.
http://ag1le.blogspot.com.au/2011/12/antenna-experiments-fractal-quad-for-28.html

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PYTHAGORAS FLIRTS WITH LUMERACY AND TECHNOLOGY IN THE AEGEAN SEA

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The presentation will demonstrate how the Pythagorean Theorem can be presented to middle and upper school students with flair. It uses Lumeracy resources with the aid of contemporary computer algebra system based technology working in tandem with connected classrooms with wireless communication. This is a rich task lesson plan for teachers to use in classrooms. It is cross-curricula oriented and spreads its wings into the learning areas of English language, history and geography which in turn contextualise the theorem. This is not just change but a paradigm shift in mathematics pedagogy.

Aims of the Australian Curriculum

The *Australian Curriculum: Mathematics* aims to ensure that students:

- are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens
- develop an increasingly sophisticated understanding of mathematical concepts and fluency with processes, and are able to pose and solve problems and reason in Number and Algebra, Measurement and Geometry, and Statistics and Probability
- recognise connections between the areas of mathematics and other disciplines and appreciate mathematics as an accessible and enjoyable discipline to study (Australian Curriculum Assessment and Reporting Authority, 2012).

Rationale of the Australian Curriculum

Learning mathematics creates opportunities for and enriches the lives of all Australians. The Australian Curriculum: Mathematics provides students with essential mathematical skills and knowledge in *Number and Algebra, Measurement and Geometry, and Statistics and Probability*. It develops the numeracy capabilities that all students need in their personal, work and civic life, and provides the fundamentals on which mathematical specialties and professional applications of mathematics are built.

General Capabilities

In the Australian Curriculum, the general capabilities encompass the knowledge, skills, behaviours and dispositions that, together with curriculum content in each learning area and the cross-curriculum priorities, will assist students to live and work successfully in the twenty-first century.

There are seven general capabilities:

- Literacy
- Numeracy
- Information and communication technology (ICT) capability
- Critical and creative thinking
- Personal and social capability
- Ethical understanding
- Intercultural understanding.

Intercultural Understanding and History

Students develop intercultural understanding as they learn to value their own cultures, languages and beliefs, and those of others. They come to understand how personal, group and national identities are shaped, and the variable and changing nature of culture. The capability involves students in learning about and engaging with diverse cultures in ways that recognise commonalities and differences, create connections with others and cultivate mutual respect.

Learning in history includes interpreting and representing large numbers and a range of data such as those associated with population statistics and growth, financial data, figures for exports and imports, immigration statistics, mortality rates, war enlistments and casualty figures; chance events, correlation and causation; imagining timelines and time frames to reconcile related events; and the perception and spatial visualisation required for geopolitical considerations, such as changes in borders of states and in ecology.

The Lesson Plan

1. Though heuristic learning is essential in mathematics teaching, it is difficult for students in year nine (9) the learning year we introduce Pythagorean Theorem, to understand let alone discover such an elegant concept.
2. The lesson should explicitly introduce, by students drawing a 3, 4, 5 Pythagorean triplet right angle triangle and investigating the squares of each side and the sum of two short side squares.
3. Consolidate the concept by some more triplets to satisfy them with the truth of the theorem with inductive reasoning. CAS machine will come in handy as this investigation can be completed quickly on the machine.
4. This should be followed by Lumeracy material in the form of "Darth Vader video" and the reading of the book "What is your angle Pythagoras?" (Ellis, undated)
5. Technology is then brought into play by wirelessly sending the map of the Aegean Sea showing Crete, Rhodes and Samos narrated in the book. The three islands form a right angled triangle (almost) and the theorem is used to find the distance between Crete and Samos with the use of CAS and connected classroom technology.
6. Next introduce the students to deductive reasoning using the available geometrical proofs.
7. This should follow practical examples in construction, land survey etc...
8. Throughout the lessons cross curricula material from geography, history and English language strands should be discussed. The lesson also should weave all strands of the Australian Curriculum: Mathematics. Missing in this is the Statistics and Probability strand, easily included by discussing the probabilities of successful crossing of the Aegean and statistical work on different routes of travel and the mean distance of Crete to Samos.

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Mathematics
of Planet Earth 2013

- Student Paper

DO ALGEBRAIC LETTERS CONTINUE TO REPRESENT OBJECTS OR WORDS FOR OUR STUDENTS?

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Comprehension of worded problems is a major difficulty for students in a majority of Australian schools. Many students can reproduce algorithms taught in class, but cannot solve similar tasks given as worded problems. For this study, worded problems refer to mathematical problem solving questions expressed in narrative form, which are solved either arithmetically or algebraically. The study investigated the use of journal writing to help students improve their problem solving skills and demonstrate their mathematical reasoning and understanding to their teachers. A major misconception identified in journal writing responses involved students incorrectly interpreting that algebraic letters stand for objects or something starting with that letter, which evokes the concept of fruit salad algebra. And, this often led to students experiencing difficulty solving worded problems.

Background

This research took place in a single-sex independent school situated in the northern suburbs of a major Australian city. This school was selected because it has a significant population of students from various ethnic backgrounds, where the majority speak additional languages other than English at home. Seventeen Year 10 advanced mathematics students (aged 15-16 years) participated in this study.

Introduction

Many students in Australian schools have literacy issues in mathematics; in particular, they struggle with understanding worded problems (COAG, 2008; White, 2010). For many students mathematics is like a 'foreign language', symbols, expressions and even some words could be challenging barriers that could inhibit them from understanding mathematical concepts (COAG, 2008, p. 32). The Concepts in Secondary Mathematics and Science [CSMS] research project (Hart, 1981) concluded that most students up to the age of 15 are "unable to interpret algebraic letters as generalized numbers or even as specific unknowns" (MacGregor & Stacey, 1997, p. 1). The Australian Curriculum recognises the significance of problem solving in mathematics classrooms by including it as one of four proficiency strands in the Australian Curriculum: Mathematics (ACARA, 2013). However, students spend most of their time working through procedural problems reproducing algorithms (Nosegbe-Okoka, 2004), hence, they lack skill, interest and enthusiasm to attempt and solve authentic worded problems (Shulman & Armitage, 2005). Therefore, this study intended to investigate how journal writing can help students improve their problem solving skills and strategies to enlighten the emerging understandings of how writing is related to problem solving.

Journal Writing

The notion of journal writing in mathematics classrooms was established in the 1980s (Borasi & Rose, 1989; Vukovich, 1985). The majority of the literature drew attention to how journal writing could be used by teachers to structure mathematical writing and encourage students to construct mathematical knowledge and develop communication skills (Borasi & Rose, 1989; Clarke, Waywood, & Stephens, 1993; Lim & Pugalee, 2004; Waywood, 1992;). Journal writing can take many forms and generally is classified into one of three categories: a restatement of information; a summary of information; or a mode of dialogue with the teachers (Waywood, 1992). It mainly involves describing step-by-step how to solve problems or reflect on daily lessons (Clarke et al., 1993; Lim & Pugalee, 2004). However, journal writing in this study required students to explain concepts, apply their knowledge, and write and reflect on what they had done including how they had solved worded problems. Students were required to answer a series of questions, which acted as prompts. For example,

- What does the problem ask you to find?
- What information do you need to solve this problem?
- How did you solve the problem? (Show your working out and explain what you did).

- If you correctly solved the problem, explain why you think your solution is correct.
- If you incorrectly solved the problem, explain why you think your solution is incorrect. Where may have you made a mistake?
- What have you learnt from completing the problem? Was there anything you didn't understand? (e.g., words).

Written Communication in Journal Writing to Improve Problem Solving

Writing in schools is valuable for students to demonstrate what has been learnt. It enables students to make their thinking visible and confront their own understanding of mathematical ideas (Whitin & Whitin, 2002). Pugalee (2005) stated that writing supports students with their mathematical reasoning and problem solving. Writing also "allows teachers to give feedback to students to assess their learning process" (Ng, 2007, p. 1). Therefore, mathematics classrooms need to use writing to assist students to make connections between what they read, hear, think and understand. It was anticipated that journal writing would help teachers to encourage students to structure mathematical writing, develop communication skills, deepen their mathematical comprehension and improve their mathematical literacy (Clarke et al., 1993; Lim & Pugalee, 2004; Waywood, 1992; Countryman, 1992; Borasi & Rose, 1989). Thus, mathematics teachers need to make important considerations and reflect on how they could develop classroom structures that encourage different teaching styles to facilitate student learning. They will also need to think about how they could support students to improve their problem solving skills and strategies to become better and more confident with worded problems.

Students' Misconceptions about Algebraic Notation

While students are learning algebra or solving algebraic worded problems it is important for them to learn and understand that algebraic letters represent numbers or specific unknowns. A study by Macgregor and Stacey (1997) revealed that students between the ages of 11-15 share common misunderstandings. Students interpreted algebraic letters in many ways such as;

- Letter ignored, i.e., If I add 5 to $p + 1$, I get 6.
- Numerical value, i.e., a as 1, b as 2 etc.
- Abbreviated word, i.e., *Uh* as "unknown height".
- Alphabetical value, i.e., $h = 8$ because it is the 8th letter of the alphabet.
- Use different letters for each unknown, i.e., choosing g to denote the height of an object.

- Unknown quantity, i.e., $p = 7$, for $p + q = 12$ and p is a natural number greater than q.
- Letter is a label associated with an object, i.e., a to mean apples.
- Letters equal 1, i.e.. $10 + h = 11$
- Letter has a general referent, i.e., h mean "height", so it means both "David's height" and "Con's height" in the statement $h = h + 10$.

Some of these were an extension to Küchemann's (1981) classification of students' interpretation of algebraic letters. However, for this study, there was only one common misconception evident in student journals: interpreting letters as an object. Chick (2009) describes this as "fruit salad algebra", where, a is for apples and b is for bananas, which she found present in some textbooks. This was not the case for the textbooks used in this classroom (Year 10 ICE-EM Mathematics, 2011 reference). Over six weeks there were 11 sessions, which each took approximately 25 minutes of class time. Once the data was collected, the researcher and teacher spent additional time perusing more deeply students' journal writing responses i.e. analysing students' problem solving strategies and how they defined the algebraic letters. This helped them analyse whether the journals indicated signs of improvement in students' problem solving skills and strategies. While students portrayed improvements in their problem solving skills and strategies, they seemed to have trouble representing algebraic letters with their intrinsic meaning i.e., for a particular problem some students wrote, 'x = adults' rather than 'x = the number of adult tickets sold' as they interpreted letters as an object. This did not cause difficulty or confusion when students were dealing with simple algebraic worded problems. However, as the level of difficulty of the problems increased, student journals demonstrated that many of them experienced difficulties with comprehending the problem, which was mainly related to the *fruit salad algebra* misconception.

Interpreting Letter as an Object

The *fruit salad algebra* misconception was common among the students, where algebraic letters were interpreted as objects when they were solving worded problems in their journals. This is one of few prominent misconceptions described by Küchemann (1981), where a letter, "rather than clearly being a placeholder for a number, is regarded as being an object (the letter is a label associated with an object i.e. a for apples is such an example)" (Chick, 2009, p.1). During the study, some students developed the habit of correctly interpreting letters as an unknown number, whereas, others continued to portray this type of thinking when they were solving worded problems. 'Mary's age problem' was one of the difficult problems for students. For example,

> Mary is 3 times as old as her son. In 12 years, Mary's age will be 1 year less than twice her son's age. How old is each now?

A majority of the students portrayed the traits of *fruit salad algebra* misconception when they were defining the letters. Many students wrote "M = Mary's age" and "S = Son's age", when in fact, they should have defined both letters in terms of their *ages now*, which caused a lot of confusion when students tried to set up equations. Most students wrote the equations for Mary's age in *12 years*, as "$M + 12 = 2S - 1$", which was incorrect because both "M" and "S" represented their ages now, therefore, the equations did not reflect Mary's and her son's age in 12 years. Students who wrote this still got the correct solution, but for the wrong reasons. They were not able to interpret that "$2S$" represented two times son's age now. As indicated by MacGregor and Stacey (1997), students usually learn to denote concepts by the "initial letters of their names (A for area, m for mass, t for time, etc.)" (p. 24). The way these letters were used most probably supported students' belief that letters in algebra could represent objects or words instead of numbers.

Only one student (*see Figure 1*) was able to correctly define the letters in terms of *Mary's and her son's age now* and set up the equation as "$M + 12 = 2(S + 12) - 1$". This indicated that he understood that "$2(S + 12)$" represent two times the *son's age in 12 years*. This particular student continued to demonstrate this behavior in his other journal writing responses.

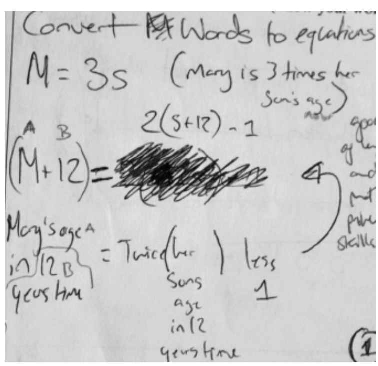

Figure 1. An example of a student who correctly defined algebraic letters and set up equations to solve Mary's age problem.

Over the course of the journal writing practice, most students showed improvements in their problem solving skills and strategies. However, some students were persistent in interpreting letters as objects. 'Roberto's problem', was correctly solved by a majority of the students and many showed improvements in their definition of letters. For example,

> Roberto went to a big brand factory outlet and he spent $61 dollars on a new blazer. The cost of the blazer was $15 less than two times what he spent on a trouser. How much did he spend on a trouser?

Most students wrote, "x = the cost of the blazer … t = the amount spent on a trouser", which revealed that they did not associate letters with objects, but recognized that letters represent an amount of something or an unknown number. However, some students (*see Figure 2*) continued to write, 'x = Blazer, y = Trouser…b = blazer, x = trouser', which indicated that they were not understanding that letters do not represent objects or words. Even though some of these students achieved the correct solution, a few students could not solve the question correctly.

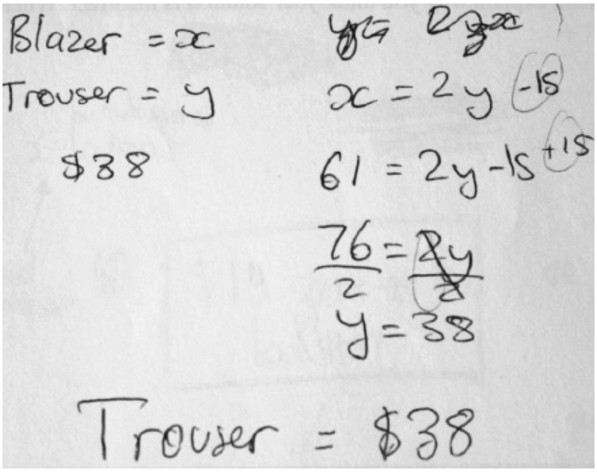

Figure 2. An example of a student portraying the fruit salad algebra misconception and getting a correct solution.

Some students continued to interpret letters as objects in 'Karl's money problem'.

> Karl divided $15.20 among 3 people so that the second will have one dollar more than the first, and the third will have $2.70 more than the second. Find how much each person gets.

Only half of the students solved this question correctly using appropriate problem solving skills and strategies, however, *fruit salad algebra* misconceptions were prevalent. Many students wrote "First person = x, second person = x + 1…" whereas, x should have been defined as *the amount of money the first person receives*. This indicated that students do not quite understand the meaning of letters in algebra, even though some students who interpret letters as objects may still solve the worded problems correctly. In the long run, for more challenging questions such as *age problems* or *money problems*, if students believe that "the letter must stand for something beginning with that letter [fruit salad algebra]…[it] leads to major problems" (Hathaway & Prasad, 1994, p. 91). This will also lead to students experiencing "difficulties in making sense of algebra" (MacGregor & Stacey, 1997, p. 26) and getting confused when they are solving more difficult worded problems.

One student (*see Figure 3*) provided a much clearer definition of the algebraic letters as he explicitly wrote and showed what "$1 more than the first person" and "$2.70 more than the second person" meant. This indicated he was aware of the language skills involved in writing algebraic expressions and what the letters represented.

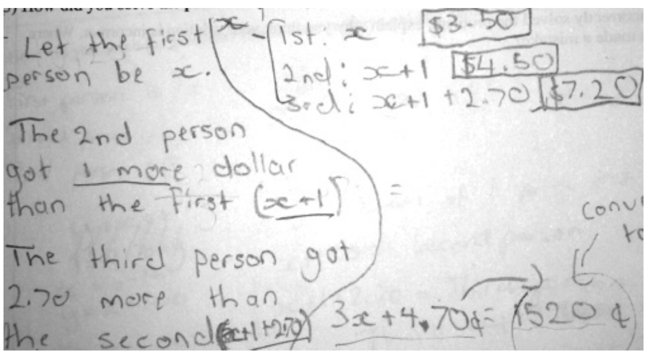

Figure 3. An example of a student's journal indicating his awareness of the language skills involved in writing algebra expressions.

Conclusion

In this study, journal writing appeared to help students improve their problem solving skills and strategies. As students practised journal writing their written communication skills improved, however, some students continued to interpret letters as objects or words (*fruit salad algebra*). Other students developed habits of avoiding *fruit salad algebra*, but it appears that, in general, students did not understand the importance of representing algebraic letters "as generalised numbers or specific unknowns" (MacGregor & Stacey, 1997, p. 1). During one of the class discussions about a worded problem, one student persisted with the belief that defining letters as objects or words does not really matter. The researcher had to intervene and discuss with the class reasons as to why letters as objects is a misleading approach.

The student journal responses appear to confirm previous research from Chick (2009) that suggested "fruit salad algebra approach is well-entrenched in the teaching culture (from previous years) ... because that is how teachers themselves were taught" (p. 7). But the classroom teacher indicated that he does not encourage the *fruit salad algebra* approach.

Teachers need to assist students to improve their problem solving skills and strategies. This includes reinforcing that algebraic letters represent generalised numbers or specific unknowns. Activities such as journal writing could be used in classrooms to help teacher achieve these outcomes.

MacGregor and Stacey (1997) suggest that students interpreting letters as objects may result from certain "concepts in applied mathematics ... usually [being] denoted by the initial letters of their names" (p. 24). This could be reinforcing the belief that letters in expressions or formulas represent objects or words. As a result, Chick (2009) stated that there may well be teachers who practice the *fruit salad algebra* approach because it is "perceived by teachers as being accessible for students" (p. 7). Therefore, the aim for teachers would be to recognise and address this misconception to eliminate chances of these misconceptions either occurring or continuing in mathematics classrooms. This also appears to inhibit students from correctly solving worded problems. Teachers must be wary of the beliefs that letters represent objects and convince themselves and their students of the potential disadvantages of *fruit salad algebra* (Chick, 2009).

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